Conversion of Matrix ODEs to Certain Universal and Easily Handlable Forms Via Space Extension

SEVDA ÜSKÜPLÜ
İstanbul Technical University
Informatics Institute
Ayazağa Campus, 34469, Maslak, İstanbul
TÜRKİYE

METİN DEMİRALP
İstanbul Technical University
Informatics Institute
Ayazağa Campus, 34469, Maslak, İstanbul
TÜRKİYE

Abstract: This paper is focused on the conversion of matrix ordinary differential equations to certain universal forms which can be handled more easily than their original structures when the solutions to them is sought. This goal is realised by using the space extension concept. The concerned equations are chosen from certain specific ones although the generalization seems to be possible for all types of matrix ODEs. This paper concerns with the obtaining of new universal form not its solution. However certain clues about the solution technique construction are also presented.

Key–Words: Ordinary Differential Equations, Space Extension Concept, Matrices.

1 Introduction

Space extension concept is not a new issue. It is used, for example, in the order reduction of ordinary or partial differential equations. An, say n–th order ODE with a scalar unknown function can be converted to n first order ODEs by considering first $(n-1)$ derivative of the unknown scalar function $(n-1)$ last unknown components of an n–element vector whose first element is the unknown function's itself [1].

On the other hand, any set of first order ODEs which are expressing the derivatives of the unknowns in terms of certain nonlinear functions of the independent variable and the unknown functions through separate equations can be reduced to same type of equations but with at most quadratic nonlinearity as long as the nonliner function's gradients with respect to unknowns remain in the same family of functions [2].

Quite recently we have also developed a matrix ODE solver based on the space extension concept which finds its roots in the following lines of this section [3].

It seems to be better to give certain details of the space extension concept before we proceed to get the universal form mentioned in the title of this paper. There are several ways to explain what the space extension concept is. Amongst these, perhaps, the most inductive one is based on the decomposition of a given function of a complex variable to odd and even components. If we consider a function analytic in every finite domain of the complex plane of its argument then we can express it as an infinite linear combinations of the natural number powers of its argument, that is,

$$f(z) \equiv \sum_{j=0}^{\infty} f_j z^j \tag{1}$$

where f_j coefficient is the value of j–th order derivative of the function at the origin over $j!$. This is very well known as Taylor (or more specifically McLaurin) series expansion. If we separate out this representation to an even and odd function which are denoted by $f_E(x)$ and $f_O(x)$ respectively then the following equalities can be written

$$\begin{aligned} f_E(z) &\equiv \sum_{j=0}^{\infty} f_{2j} z^{2j}, \\ f_O(z) &\equiv \sum_{j=0}^{\infty} f_{2j+1} z^{2j+1} \end{aligned} \tag{2}$$

These function do not behave equivalently when the z–plane is rotated π radians, $f_O(z)$ changes its sign while $f_E(z)$ remains unchanged. Whereas it is better to deal with two functions both of which behave equaivalently under this rotation. The sign change in $f_O(x)$ comes from the odd powers of z and can be prevented by extracting z from $f_O(z)$. This urges us to define and use the following functions

$$f_1(z^2) \equiv \sum_{j=0}^{\infty} f_{2j} z^{2j},$$

$$f_2(z^2) \equiv \frac{f_O(z)}{z} \equiv \sum_{j=0}^{\infty} f_{2j+1} z^{2j+1} \tag{3}$$

instead of the ones defined in (1). Then we can rewrite (1) as follows

$$f(z) \equiv f_1(z^2) + z f_2(z^2) \tag{4}$$

We could of course assume the right hand structure with the unknown–yet–functions depending not on z but its square without considering what we have discussed above. If we would do so then we would need to impose a rule for the determination of these entities. Since one rule is (3)'s itself just a single new rule would suffice. As we can easily see the replacement of z by $-z$, which is corresponding to a π radian rotation, leaves these unknown functions unchanged as long as the original function does not have any branch point singularity at the origin of z–plane. However, their linear combination coefficients in (3) are affected by this transformation and we obtain the following equation from (3) after this rotation

$$f(-z) \equiv f_1(z^2) - z f_2(z^2) \tag{5}$$

(3) together with (4) presents us a linear algebraic equation set of two unknowns, $f_1(z^2)$ and $f_2(z^2)$. Their solution are obtained as follows

$$\begin{aligned} f_1(z^2) &= \frac{f(z)+f(-z)}{2}, \\ f_2(z^2) &= \frac{f(z)-f(-z)}{2z} \end{aligned} \tag{6}$$

which are in harmony with what we expect to get.

In obtaining (6) from the assumption in (4) does not require the analyticity of $f(z)$ function. Laurent type series, that is, the functions with polar singularities can be equivalently tackled with. This is because of polar type singularities do not affect the angle of π radian rotations. On the contrary, branch point singularities do it. This is the reason why we impose the nonexistence of branch points singularities in the function.

The equation given by (3) can be rewritten in matrix notation as follows

$$f(z) = \mathbf{z}^T \mathbf{f}(z^2) \tag{7}$$

where

$$\begin{aligned} \mathbf{z}^T &\equiv \left[\, 1 \; z \,\right], \\ \mathbf{f}(z^2)^T &\equiv \left[\, f_1(z^2) \; f_2(z^2) \,\right], \end{aligned} \tag{8}$$

If we try to make an analogy to the trivial equality $f(z) \equiv 1 f(z)$ then we can see the generalization. Although (7) is a scalar equality its multiplicants are replaced by two element vectors, $\mathbf{z}^t$ (a row vector) for 1 and $\mathbf{f}(z^2)$ (a column vector) for $f(z)$. Therefore the scalar entities which can be considered one element vectors are replaced now by two element vectors. That is the space is extended from 1 dimension to 2 dimensions. The important things in (7) are the lonely zeroth and first powers of z as the linear combination coefficients and the argument z^2 which enables us to use π radian rotation. The two consecutive applications of this rotation puts the rotated plane to its original position as long as the original function has no branch point singularity.

Until now we tackled with the space extension from one dimension to two dimensions. Everything here are not peculiar only to this limited cases. On the contrary, they can be equivalently applied to space extension from one dimension to any finite or denumerably infinite dimension. For instance we can write the space extension equality from one dimension to three dimension as follows

$$f(z) = \mathbf{z}^T \mathbf{f}(z^2) \tag{9}$$

where

$$\begin{aligned} \mathbf{z}^T &\equiv \left[\, 1 \; z \; z^2 \,\right], \\ \mathbf{f}(z^3)^T &\equiv \left[\, f_1(z^3) \; f_2(z^3) \; f_3(z^3) \,\right], \end{aligned} \tag{10}$$

and we assumed that $f(z)$ has no branch point singularity. Here the z^3 argument enforces us to use $2\pi/3$ radian rotation to leave the argument unchanged. Also, its twice consecutive application leaves the z^3 argument unchanged. Therefore the governing equations for the determination of $f_1(z^3)$, $f_3(z^3)$, and $f_3(z^3)$ functions can be constructed as follows after using these rotations appropriately

$$\begin{aligned} &\begin{bmatrix} 1 & z & z^2 \\ 1 & z e^{i\frac{2\pi}{3}} & z^2 e^{i\frac{4\pi}{3}} \\ 1 & z e^{i\frac{4\pi}{3}} & z^2 e^{i\frac{2\pi}{3}} \end{bmatrix} \begin{bmatrix} f_1\left(z^3\right) \\ f_2\left(z^3\right) \\ f_3\left(z^3\right) \end{bmatrix} \\ &= \begin{bmatrix} f(z) \\ f\left(z e^{i\frac{2\pi}{3}}\right) \\ f\left(z^2 e^{i\frac{4\pi}{3}}\right) \end{bmatrix} \end{aligned} \tag{11}$$

where the coefficient matrix of the unknown vector has nonvanishing Vandermonde type determinant and therefore can be inverted uniquely allowing us to get unique solution.

The equation above can be generalised to n dimension easily. In that general case all exponential terms will have n in their denominators instead of 3 and their all less than n natural number powers will appear appropriately.

The rest of the paper is organised as follows. The second section presents the matrix ODEs which will be used as targets for the application of the space extension concept. The third section is devoted to the construction of universal and easily handlable equations after the space extension is applied on the target equatio. Fourth section finalizes the paper with concluding remarks.

2 Target Matrix ODEs for Space Extension

Perhaps the most easily handlable structures amongst the matrix ordinary differential equations are as follows

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t), \tag{12}$$

$$\dot{\mathbf{Y}}(t) = \mathbf{Y}(t)\mathbf{B}(t), \tag{13}$$

where dot stands for the differentiation with respect to t, $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are given square matrix functions of t, and $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ represent the unknown square matrix functions of t same type with $\mathbf{A}(t)$ and $\mathbf{B}(t)$ respectively.

The construction of a solution to these equations depend on the structures of the given t dependent entities and contain an arbitrary matrix which can not be uniquely determined unless an appropriate condition is imposed. For example if the one or both of the left or right coefficient matrices are constant the corresponding solution(s) appears to be, an exponential matrix with a t proportional matrix argument, multiplied by an arbitrary constant matrix from left or right depending on whether the equation is (12) or (13).

If the case under consideration is not this then series solutions can be constructed depending on the singularities of the relevant matrix ODE. If the singularity is regular one then Frobenius type series solutions can be constructed and converge within a fixed disc of t–complex plane. The infinite convergence radius may be encountered if the singular points are nowhere but at infinity. Otherwise, if there is an isolated singularity or in other words an essential singularity then only asymptotic solutions can be generated.

In some of our previous works we treat these matrix ODEs via space extension and we could be able to produce certain universal equations matching the equations first investigated by Okubo. Then by defining appropriate approximation schemes for these equations we could be able to produce good quality approximant for the representation of the solutions to these equations.

In this work we go to one step beyond these results and confine ourselves to the investigations of the following, a little bit complicated form of equations

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t) \tag{14}$$

where dot stands for the differentiation with respect to t again, $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are given square matrix functions of t, and $\mathbf{X}(t)$ represents the unknown square matrix function of t same type with $\mathbf{A}(t)$ and $\mathbf{B}(t)$. The solution of this equation can be analytically constructed when $\mathbf{A}(t)$and $\mathbf{B}(t)$ are both constants. In that case the product of a constant matrix from left by $e^{t\mathbf{A}}$ and from right by $e^{t\mathbf{B}}$ where $\mathbf{A}$ and $\mathbf{B}$ stand for the constant values of $\mathbf{A}(t)$ and $\mathbf{B}(t)$. The variant coefficient cases bring a lot of complications depending on how singular $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are and can be solved by using certain standard techniques like series solutions.

Here our goal is not to construct a solution of this type matrix ODEs. Instead we focus on the construction of block matrix elemented matrix ODEs which can be handled to develop good quality approximations to the solutions.

3 Space Extension on Target ODEs

In this section, a new form is constructed for the equation (14). Under the assumption that $\mathbf{T}(t)$, $\mathbf{Y}(t)$ and $\mathbf{U}(t)$ are matrix valued unknown functions, the following equality can be written:

$$\mathbf{X}(t) = \mathbf{T}(t)\mathbf{Y}(t)\mathbf{U}(t) \tag{15}$$

If $\mathbf{T}(t)$ and $\mathbf{U}(t)$ are invertible matrices and equation (15) is substituted into equation (14) the following equation is obtained.

$$\begin{aligned}\dot{\mathbf{Y}}(t) &= \mathbf{T}(t)^{-1}\left(\mathbf{A}(t)\mathbf{T}(t) - \dot{\mathbf{T}}(t)\right)\mathbf{Y}(t) \\ &\quad +\mathbf{Y}(t)\left(\mathbf{U}(t)\mathbf{B}(t) - \dot{\mathbf{U}}(t)\right)\mathbf{U}(t)^{-1}\end{aligned} \tag{16}$$

Equation (16) has three unknown matrices and it is possible to choose two of them arbitrarily. If this choice is made as it will make the right side equal to zero, without depending on the choice of $\mathbf{Y}(t)$, the following independent equations for $\mathbf{T}(t)$ and $\mathbf{U}(t)$ functions can be obtained.

$$\dot{\mathbf{T}}(t) = \mathbf{A}(t)\mathbf{T}(t), \qquad \dot{\mathbf{U}}(t) = \mathbf{U}(t)\mathbf{B}(t) \tag{17}$$

The space extension method can be applied to both these equations in (17). Under these circumstances, the following equality can be written

$$\dot{\mathbf{Y}}(t) = \mathbf{0} \tag{18}$$

and, if it is solved, then

$$\mathbf{Y}(t) = \mathbf{K} \tag{19}$$

is obtained where K is a constant square matrix. If equation (19) is substituted into equation (15) then

$$\mathbf{X}(t) = \mathbf{T}(t)\mathbf{K}\mathbf{U}(t) \tag{20}$$

is obtained. Since $\mathbf{T}(t)$ and $\mathbf{U}(t)$ matrices can be solved from equations (17), $\mathbf{X}(t)$ function can be also solved.

In equation (14) the coefficient matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ can be chosen as polynomials of any order as following:

$$\mathbf{A}(t) = \mathbf{A}_0 + t\mathbf{A}_1 + \cdots + t^{m-1}\mathbf{A}_{m-1} \tag{21}$$

$$\mathbf{B}(t) = \mathbf{B}_0 + t\mathbf{B}_1 + \cdots + t^{m-1}\mathbf{B}_{m-1} \tag{22}$$

If these equations are substituted into equations (17) and they are equaled to zero then we obtain m equations which contain $\mathbf{T}_0, \mathbf{T}_1, \cdots, \mathbf{T}_{m-1}$ and m equations which contain $\mathbf{U}_0, \mathbf{U}_1, \cdots, \mathbf{U}_{m-1}$. Using these equations, it is possible to define some new unknowns and form new differential equations. For this reason, following definitions can be made

$$\mathbf{X}_{j\,k}(y) \equiv \mathbf{T}_j(y)\mathbf{K}\mathbf{U}_k(y), \quad j,k = 0,1,\cdots,m-1 \tag{23}$$

If the derivatives of these $m \times m$ equations are taken, and, the derivatives of $\mathbf{T}_j$'s and $\mathbf{U}_j$'s are substituted into obtained equations and after some modifications, m equations which contain just $\mathbf{X}_{j\,k}$ matrices and its derivatives are obtained

$$\begin{aligned}\mathbf{X}'_{jk} &= \frac{1}{y}\Big(\sum_{i=0}^{j-1}\mathbf{A}_{j-1-i}\mathbf{X}_{ik} \\ &+ \sum_{i=0}^{k-1}\mathbf{X}_{ji}\mathbf{B}_{k-i-1}\Big) \\ &+ \Big(\sum_{i=0}^{m-j-1}\mathbf{A}_{m-i-1}\mathbf{X}_{i+jk} \\ &+ \sum_{i=0}^{m-i-1}\mathbf{X}_{jm-i-1}\mathbf{B}_{i+k}\Big)\end{aligned} \tag{24}$$

where $j,k = 0,\cdots,m-1$. In order to give a more understandable form to these equations the following blocked matrix can be written

$$\Xi(\mathbf{y}) \equiv \begin{bmatrix} \mathbf{X}_{00}(y) & \mathbf{X}_{01}(y) & \cdots & \mathbf{X}_{0m-1}(y) \\ \mathbf{X}_{10}(y) & \mathbf{X}_{11}(y) & \cdots & \mathbf{X}_{1m-1}(y) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{X}_{m-1\,0}(y) & \mathbf{X}_{m-1\,1}(y) & \cdots & \mathbf{X}_{m-1\,2m-1}(y) \end{bmatrix} \tag{25}$$

Using this matrix which depends on variable y, after making some modifications the $m \times m$ equations can be written as one matrix valued differential equation.

$$\begin{aligned}\Xi'(\mathbf{y}) &= \tfrac{1}{y}\left(\mathcal{A}_{-\mathbf{1}}\Xi(\mathbf{y}) + \Xi(\mathbf{y})\mathcal{B}_{-\mathbf{1}}\right) \\ &\quad + \left(\mathcal{A}_{\mathbf{0}}\Xi(\mathbf{y}) + \Xi(\mathbf{y})\mathcal{B}_{\mathbf{0}}\right)\end{aligned} \tag{26}$$

where

$$\mathcal{A}_{-\mathbf{1}} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \frac{1}{m}\mathbf{A}_0 & -\frac{1}{m}\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m}\mathbf{A}_{m-1} & \frac{1}{m}\mathbf{A}_{m-2} & \frac{1}{m}\mathbf{A}_{m-3} & \cdots & -\frac{m-1}{m}\mathbf{I} \end{bmatrix} \tag{27}$$

$$\mathcal{A}_{\mathbf{0}} \equiv \begin{bmatrix} \frac{1}{m}\mathbf{A}_{m-1} & \frac{1}{m}\mathbf{A}_{m-2} & \frac{1}{m}\mathbf{A}_{m-3} & \cdots & \frac{1}{m}\mathbf{A}_0 \\ \mathbf{0} & \frac{1}{m}\mathbf{A}_{m-1} & \frac{1}{m}\mathbf{A}_{m-2} & \cdots & \frac{1}{m}\mathbf{A}_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \frac{1}{m}\mathbf{A}_{m-1} \end{bmatrix} \tag{28}$$

$$\mathcal{B}_{-\mathbf{1}} \equiv \begin{bmatrix} \mathbf{0} & \frac{1}{m}\mathbf{B}_0 & \frac{1}{m}\mathbf{B}_1 & \cdots & \frac{1}{m}\mathbf{B}_{m-2} \\ \mathbf{0} & -\frac{1}{m}\mathbf{I} & \frac{1}{m}\mathbf{B}_0 & \cdots & \frac{1}{m}\mathbf{B}_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\frac{m-1}{m}\mathbf{I} \end{bmatrix} \tag{29}$$

$$\mathcal{B}_{\mathbf{0}} \equiv \begin{bmatrix} \frac{1}{m}\mathbf{B}_{m-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \frac{1}{m}\mathbf{B}_{m-2} & \frac{1}{m}\mathbf{B}_{m-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m}\mathbf{B}_0 & \frac{1}{m}\mathbf{B}_1 & \frac{1}{m}\mathbf{B}_2 & \cdots & \frac{1}{m}\mathbf{B}_{m-1} \end{bmatrix} \tag{30}$$

These equations are similar to the Okubo Form [2, 3, 4]. Here, it is striking that the coefficient matrices are constant and this enables us to write a recursive relationship with two terms by using series solutions. Under this consideration, this equation can be called as "Universal Form With Two Consecutive Term Recursion".

4 Concluding Remarks

The main goal of this work has been the testing of the applicability of the space extension method on ordinary matrix differential equations. In this direction they have some preliminary works. However the chosen mathematical structure of the systems in those works were rather simple and limited. Here we tried to extend the situation to a little bit complicated cases. We arrived at what we expect in fact. This encourages us to extend our method to more complicated and more general cases. Perhaps, even nonlinear systems may be involved in the application area of space extension method. Here we have not intended to solve the resulting equations since our main purpose was to get the universal form only.

Acknowledgements: First and second authors are grateful to State Planning Agency of Turkey and Turkish Academy of Sciences respectively for their supports.

References:

[1] M. Demiralp, and, H. Rabitz, *Factorization Of Certain Evolution Operators Using Lie Algebra: Formulation Of The Method*, Journal Of Mathematical Chemistry, **6**, 1991, pp. 165 - 191.

[2] M. Demiralp, *Conversion of The First Order Linear Vector Differential Equations With Polynomial Coefficient Matrix To Okubo Form*, Proceedings of The 11th WSEAS Int. Conf. on Applied Mathematics, Dallas, Texas, USA, March 22-24, 2007.

[3] S. Üsküplü, M. Demiralp *Transformation of Ordinary Differential Equations Into Okubo Universal Form With Space Extension, And, Its Truncation Approximations* Proceedings of The Int. Conf. on Numerical Analysis and Applied Mathematics, Corfu, Greece, October 16-21, 2007.

[4] K. Okubo, *A Global Representation of A Fundamental Set of Solutions and a Stokes Phenomenon For A System of Linear Ordinary Differential Equations*, J. Math. Soc. Japan, **1**5, 3, 1963.