A Generating Function and Some Recurrence Relations for a Family of Polynomials

RABIA AKTAS and ABDULLAH ALTIN
Ankara University
Faculty of Science
Department of Mathematics
06100 Tandogan–Ankara
TURKEY

Abstract: In the present paper, we derive some families of polynomials. Some further results of these polynomials as generating function, rodrigues formula and recurrence relations are also discussed.

Key–Words: Generating function; Rodrigues formula; recurrence relation.

1 Introduction

In recent years, a great deal of progress in the field of orthogonal polynomials has seen. Some of these polynomials have been shown to be of significance in quantum mechanics and in mathematical statistics. These polynomials such as Hermite, Laguerre and Jacobi that are orthogonal polynomials over the interval (a, b) with weight function $\omega(x)$ have many useful properties. The main properties such as recurrence relation, generating function, rodrigues formula and differential equation which hold for these polynomials are seen in [2], [3], [4], [6], [7]. Moreover, different properties of orthogonal polynomials are also studied in different aspects in [1], [5].

We now consider the following polynomials:

$$\phi_{k+n(m-1)}(x) = e^{\varphi_m(x)} \frac{d^n}{dx^n} \left(\psi_k(x) e^{-\varphi_m(x)} \right) \tag{1}$$

Let $\phi_{k+n(m-1)}(x)$ be a sequence of polynomials given by the Rodrigues formula (1). $\phi_{k+n(m-1)}(x)$ is a polynomial of degree $k + n(m-1)$, $n = 0, 1, 2, ...$and $\psi_k(x)$ and $\varphi_m(x)$ are polynomials respectively of degree k and m ; $k, m = 0, 1, 2, ...$

This paper presents a generating function and some recurrence relations for the polynomial set $\phi_{k+n(m-1)}(x)$.

2 Generating Function of the Polynomial Set $\phi_{k+n(m-1)}(x)$

In this section, we find a generating function for the polynomials $\phi_{k+n(m-1)}(x)$.

From the Cauchy integral theorem for a suitable contour C, we have

$$f(x) = \psi_k(x) e^{-\varphi_m(x)} = \frac{1}{2\pi i} \oint_C \frac{\psi_k(z) e^{-\varphi_m(z)} dz}{(z-x)}$$

and by repeated differentiation we get

$$\begin{aligned} f^{(n)}(x) &= \frac{d^n}{dx^n} \left(\psi_k(x) e^{-\varphi_m(x)} \right) \\ &= \frac{n!}{2\pi i} \oint_C \frac{\psi_k(z) e^{-\varphi_m(z)} dz}{(z-x)^{n+1}}. \end{aligned}$$

We now try to sum the series

$$\sum_{n=0}^{\infty} \frac{\phi_{k+n(m-1)}(x)}{n!} t^n$$

and by substituting

$$\begin{aligned} \phi_{k+n(m-1)}(x) &= e^{\varphi_m(x)} \frac{d^n}{dx^n} \left(\psi_k(x) e^{-\varphi_m(x)} \right) \\ &= e^{\varphi_m(x)} \frac{n!}{2\pi i} \oint_C \frac{\psi_k(z) e^{-\varphi_m(z)} dz}{(z-x)^{n+1}}. \end{aligned}$$

Hence we obtain

$$\sum_{n=0}^{\infty} \frac{\phi_{k+n(m-1)}(x)\, t^n}{n!} =$$

$$\begin{aligned} & \frac{e^{\varphi_m(x)}}{2\pi i} \oint_C \frac{\psi_k(z)\, e^{-\varphi_m(z)}}{(z-x)} \\ & \times \sum_{n=0}^{\infty} \left(\frac{t}{z-x}\right)^n dz \\ = & \frac{e^{\varphi_m(x)}}{2\pi i} \oint_C \frac{\psi_k(z)\, e^{-\varphi_m(z)} dz}{z-(x+t)} \\ = & e^{\varphi_m(x)} \psi_k(x+t)\, e^{-\varphi_m(x+t)} \end{aligned}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{\phi_{k+n(m-1)}(x)\, t^n}{n!} = \psi_k(x+t)\, e^{\varphi_m(x)-\varphi_m(x+t)} \tag{2}$$

where

$$\left|\frac{t}{z-x}\right| < 1.$$

We of course assume that the point x is inside the contour C that is the real axis and the point $x+t$ should also be inside the contour C.

It is shown that from (2), a generating function $F(x,t)$ for the polynomial set $\phi_{k+n(m-1)}(x)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\phi_{k+n(m-1)}(x)\, t^n}{n!} &= \psi_k(x+t) \\ & \quad \times e^{\varphi_m(x)-\varphi_m(x+t)} \\ &= F(x,t). \end{aligned} \tag{3}$$

3 Recurrence Relations for a Special Case of $\phi_{k+n(m-1)}(x)$

If we take $\psi_k(x) = 1$ in equation (1), we have

$$\phi_{n(m-1)}(x) = e^{\varphi_m(x)} \frac{d^n}{dx^n}\left(e^{-\varphi_m(x)}\right). \tag{4}$$

From (3), the generating function for the polynomial set $\phi_{n(m-1)}(x)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\phi_{n(m-1)}(x)}{n!} t^n &= e^{\varphi_m(x)-\varphi_m(x+t)} \\ &= F(x,t). \end{aligned} \tag{5}$$

The Maclauren series of the polynomial $\varphi_m'(x+t)$ at $t=0$ is

$$\begin{aligned} \varphi_m'(x+t) &= \varphi_m'(x) + \tfrac{\varphi_m''(x)}{1!} t + ... \\ & + \tfrac{\varphi_m^{(m)}(x)}{(m-1)!} t^{m-1} \\ &= \sum_{k=1}^{m} \tfrac{1}{(k-1)!} \varphi_m^{(k)}(x)\, t^{k-1}. \end{aligned} \tag{6}$$

Differentiating each member of the generating function (5) with respect to x and using (5) and (6), we obtain

$$\sum_{n=0}^{\infty} \frac{\phi_{n(m-1)}'(x)}{n!} t^n =$$

$$\begin{aligned} & \left(\varphi_m'(x) - \varphi_m'(x+t)\right) \\ & \times e^{\varphi_m(x)-\varphi_m(x+t)} \\ = & \left(\varphi_m'(x) - \varphi_m'(x+t)\right) \\ & \times \sum_{n=0}^{\infty} \frac{\phi_{n(m-1)}(x)}{n!} t^n \\ = & \sum_{n=0}^{\infty} \frac{\varphi_m'(x)\, \phi_{n(m-1)}(x)}{n!} t^n \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{m-1} \frac{\varphi_m^{(k+1)}(x)\, \phi_{n(m-1)}(x)}{n!k!} t^{n+k} \\ = & \sum_{n=0}^{\infty} \frac{\varphi_m'(x)\, \phi_{n(m-1)}(x)}{n!} t^n \\ & - \sum_{n=k}^{\infty} \left(\sum_{k=0}^{m-1} \frac{\varphi_m^{(k+1)}(x)\, \phi_{(n-k)(m-1)}(x)}{k!\,(n-k)!}\right) t^n. \end{aligned}$$

Thus, we arrive at the following recurrence relation for the polynomials $\phi_{n(m-1)}(x)$:

$$\begin{aligned} & \phi_{n(m-1)}'(x) = \varphi_m'(x)\, \phi_{n(m-1)}(x) \\ & - \sum_{k=0}^{m-1} \tbinom{n}{k} \varphi_m^{(k+1)}(x)\, \phi_{(n-k)(m-1)}(x) \end{aligned} \tag{7}$$

$$(n \geq k\,,\, m \geq 1).$$

Differentiating two hand side of the generating function (5) with respect to t, we find

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\phi_{n(m-1)}(x)}{n!} n t^{n-1} \\ = & -\varphi_m'(x+t)\, e^{\varphi_m(x)-\varphi_m(x+t)}. \end{aligned}$$

If we take $n+1$ instead of n in the last expression and use (5) and (6), we have

$$\sum_{n=0}^{\infty} \frac{\phi_{(n+1)(m-1)}(x)}{n!} t^n =$$

$$\begin{aligned}
&-\varphi_m^{'}(x+t)\sum_{n=0}^{\infty}\frac{\phi_{n(m-1)}(x)}{n!}t^n \\
&= -\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m-1}\frac{\varphi_m^{(k+1)}(x)}{k!}t^k\right)\frac{\phi_{n(m-1)}(x)}{n!}t^n \\
&= -\sum_{n=k}^{\infty}\left(\sum_{k=0}^{m-1}\frac{\varphi_m^{(k+1)}(x)\,\phi_{(n-k)(m-1)}(x)}{k!\,(n-k)!}\right)t^n.
\end{aligned}$$

Hence we get the following recurrence relation for the polynomials $\phi_{n(m-1)}(x)$:

$$\begin{aligned}
\phi_{(n+1)(m-1)}(x) &= -\sum_{k=0}^{m-1}\binom{n}{k}\varphi_m^{(k+1)}(x) \\
&\quad \times\phi_{(n-k)(m-1)}(x) \qquad (8)
\end{aligned}$$

$$(n \geq k\,,\ m \geq 1)\,.$$

From (7) and (8), we have a recurrence relation for these polynomials,which is easily seen to be given by

$$\begin{aligned}
\phi_{n(m-1)}^{'}(x) &= \varphi_m^{'}(x)\,\phi_{n(m-1)}(x) \qquad (9) \\
&\quad +\phi_{(n+1)(m-1)}(x)
\end{aligned}$$

$$(n \geq 0\,,\ m \geq 1)\,.$$

We now obtain another recurrence relation for the polynomials $\phi_{n(m-1)}(x)$. Using derivatives of (5) with respect to x and t, we have

$$\varphi_m^{'}(x+t)\frac{\partial F}{\partial x} = \left(\varphi_m^{'}(x+t) - \varphi_m^{\prime}(x)\right)\frac{\partial F}{\partial t}. \qquad (10)$$

Writing the values of $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial t}$ in the equality (10) and using the equality (6), we find

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\frac{\phi_{n(m-1)}^{'}(x)\varphi_m^{(k+1)}(x)}{n!k!}t^{n+k} = \\
&-\sum_{n=0}^{\infty}\frac{\varphi_m^{'}(x)\phi_{(n+1)(m-1)}(x)}{n!}t^n \\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\frac{\phi_{(n+1)(m-1)}(x)\varphi_m^{(k+1)}(x)}{n!k!}t^{n+k}.
\end{aligned}$$

If we take $n-k$ instead of n in double summations in the both sides of the last expression, we have

$$\begin{aligned}
&\sum_{n=k}^{\infty}\left(\sum_{k=0}^{m-1}\frac{\phi_{(n-k)(m-1)}^{'}(x)\varphi_m^{(k+1)}(x)}{(n-k)!k!}\right)t^n = \\
&-\sum_{n=0}^{\infty}\frac{\varphi_m^{'}(x)\phi_{(n+1)(m-1)}(x)}{n!}t^n \\
&+\sum_{n=k}^{\infty}\left(\sum_{k=0}^{m-1}\frac{\phi_{(n-k+1)(m-1)}(x)\varphi_m^{(k+1)}(x)}{(n-k)!k!}\right)t^n.
\end{aligned}$$

Thus, we get the following recurrence relation,

$$\begin{aligned}
&\sum_{k=0}^{m-1}\tbinom{n}{k}\varphi_m^{(k+1)}(x)\,(\phi_{(n-k)(m-1)}^{'}(x) \\
&-\phi_{(n-k+1)(m-1)}(x)) \qquad (11) \\
&= -\varphi_m^{'}(x)\,\phi_{(n+1)(m-1)}(x)\,.
\end{aligned}$$

for

$$(n \geq k\,,\ m \geq 1)\,.$$

4 Orthogonality of the Polynomials $\phi_{n+1}(x)$

Taking $\psi_k(x) = x$ and $\varphi_m(x) = x^2$ in the equation (1), we have

$$\begin{aligned}
\phi_{n+1}(x) &= e^{x^2}\frac{d^n}{dx^n}\left(xe^{-x^2}\right) \\
&= -\frac{1}{2}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) \\
&= \frac{(-1)^n}{2}H_{n+1}.
\end{aligned}$$

Since the Hermite polynomial $H_{n+1}(x)$ of degree $n+1$, $(n = 0, 1, ...)$ is an orthogonal polynomial over the interval $(-\infty, \infty)$ with weight function $\omega(x) = e^{-x^2}$, then $\phi_{n+1}(x)$ is an orthogonal polynomial of degree $n+1$ in the same interval.

This is defined by the relations

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2}\phi_{n+1}(x)\,\phi_{m+1}(x)\,dx &= 2^{n-1}(n+1)! \\
\times\sqrt{\pi}\delta_{nm}\ ;\ n,m &= 0, 1, ...
\end{aligned}$$

where

$$\delta_{mn} = \begin{cases} 1 & , \quad m = n \\ 0 & , \quad m \neq n \end{cases}.$$

References

[1] A. Altın, E. Erkuş and M.A. Özarslan, Families of linear generating functions for polynomials in two variables. Integral Transforms and Special Functions, 17(5) (2006), 315-320.

[2] R. Askey, G.E. Andrews. and R. Roy, Special Functions, Cambridge University Press 1999.

[3] H. Hochstadt, Special Functions of Mathematical Physics, Holt, Rinehart and Winston Publications, New York 1961.

[4] E.B. McBride, Obtaining Generating Functions, Springer-Verlag, New York 1971.

[5] M.A. Özarslan, and A. Altın, Some families of generating functions for the multiple orthogonal polynomials associated with modified Bessel K-functions. Journal of Mathematical Analysis and Applications, 297, (2004) 186-193.

[6] H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions, Halsted Press, New York 1984.

[7] G. Szegö, Orthogonal Polynomials, American Mathematical Society Colloquıum Publications, Vol. 23. New York 1939.