The Sequential Frequency Assignment Process

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Abstract: A mathematical model is developed to describe the behavior of the assignment of frequencies to demands occurring one by one and distributed randomly over a space. The behavior of this Sequential Frequency Assignment Process (SFAP) is studied mainly by Monte Carlo simulations. Several conjectures concerning the long term behavior of SFAP are formulated, e.g. a law of large numbers and a central limit theorem. Also, it is shown that at higher frequencies the densities are higher than at lower frequencies. Finally, the relation of the SFAP with the car parking model and particle deposition models is discussed.

Key–Words: Frequency assignment, Almost subadditivity, Car parking problem

1 Introduction

Frequency Assignment is an important problem for radiocommunications regulatory agencies. Customers demand frequencies, e.g. for land-mobile radio communication. On one hand, the number of frequencies available is limited so that not every customer can be given a unique frequency. On the other hand, when several customers have to share a frequency, there is the risk of interference. The frequency assignment problem (FAP) is an optimization problem that attempts to assign frequencies to customers in some optimal way, e.g. minimizing the number of frequencies needed subject to restrictions regarding the interference.

Formally, the frequency assignment problem can be formulated as follows: given a set of available frequencies $\Phi = \{\phi_1, \ldots, \phi_n\}$ and a set of transmitter locations $\Xi = \{\xi_1, \ldots, \xi_m\}$, find an assignment $A : \Xi \to \Phi$ that is optimal in some sense. The FAP has been studied extensively in the past few years, mainly as a result of interest from operators of GSM systems. Numerous algorithms have been invented, see e.g. Aardal et al. [1] and Eisenblätter [2]. These algorithms address the batch frequency assignment problem, where all the locations of transmitters are fully known in advance. Operators of GSM systems and broadcast frequency planners use them to rapidly make a (near-) optimal frequency planning.

In this paper we introduce the sequential frequency assignment algorithm. Sequential frequency assignment is useful in situations when the locations where the demand for a new frequency arises are not known in advance, but arrive one at a time. This is usually the case in a large part of the land-mobile radio frequency band. In this paper, we focus on the case that assignments are irreversible, i.e. every completed assignment remains fixed during the entire process. Demand for frequencies arises sequentially, the location of the k-th customer being denoted by x_k, $k = 1, 2, \ldots$, where $x_i \in S \subset \mathbb{R}^2$. The k-th customer occupies a ball of radius δ centered in x_k. Within this ball, the frequency is not available for other customers in order to avoid radio interference. At the time of arrival of the k-th customer, only the locations $x_1, \ldots, x_k$ are known. The previous customers have already been assigned frequencies and these will not be changed.

The assignment of a frequency to the k-th customer is done according to the following algorithm:

1. Check if one of the frequencies already in use by the customers $x_1, \ldots, x_{k-1}$ is able to accommodate the k-th customer.
2. If yes, assign the lowest available frequency.
3. If no, assign a new frequency to this customer.

We refer to this algorithm as the Sequential Frequency Assignment Process (SFAP).

The assignment of a frequency and also the total number of frequencies required to accommodate customers at locations $x_1, \ldots, x_n$ depends on the order of the locations. The same set of locations arriving in a different order will generally require a different number of frequencies. In our paper we will study the model where the customers arrive according to some stochastic process $(X_k)_{k\geq 1}$. Effectively this corresponds to an average case analysis of our algorithm.

There are at least two clear advantages to this algorithm: (i) SFAP is very simple, and especially it does not require the solution of difficult and time consuming optimization procedures. (ii) SFAP assigned frequencies remain unchanged once they have been given to a customer, regardless of later demands. At the same time, it is obvious that SFAP does not distribute the frequencies in an optimal way. A batch assignment will always do a better job.

2 Mathematical Formulation

We will now give a precise mathematical formulation of the sequential frequency assignment process. In what follows, we will assume that the sequence of locations of customers demanding frequencies is the outcome of a stochastic process $(X_k)_{k\geq 1}$. Moreover we assume that the random variables X_k, $k \geq 1$, are independent and identically distributed, taking values in some bounded subset $S \subset \mathbb{R}^2$. For the mathematical approach, we assume that the set of frequencies is countably infinite. We may thus assume without loss of generality that $\Phi = \mathbb{N}^+ = \{1, 2, \ldots\}$. We denote by Φ_k the frequency assigned to the k-th customer. Clearly, Φ_k is determined by the locations of all the customers arriving up to time k, i.e. we can find a function $g_k : S^k \to \mathbb{N}^+$ such that $\Phi_k = g_k(X_1, \ldots, X_k)$. The functions g_k, $k \geq 1$, are given as follows. Consider for each integer $i \geq 1$ the set $A_{k-1}(i) := \bigcup_{1\leq j\leq k-1:\Phi_j=i} B_\delta(X_j)$, where $B_\delta(x) := \{y \in S : \|y - x\| < \delta\}$ denotes the open ball of radius δ centered at x. Thus $A_{k-1}(i)$ is the region of interference for the i-th frequency when the first $k-1$ customers have been assigned frequencies. We say that the i-th frequency is available for the k-th customer, if $B_\delta(X_k) \cap A_{k-1}(i) = \emptyset$. If one or more of the frequencies $\Phi_1, \ldots, \Phi_{k-1}$, are available, we assign the lowest available frequency to the k-th customer. Otherwise, we assign a new frequency to this customer. In this way we obtain the following formula

$$\Phi_k = \min\{i \geq 1 : B_\delta(X_k) \cap A_{k-1}(i) = \emptyset\} \quad (1)$$

Note that the process $A_k := (A_k(i))_{i\geq 1}$ defines a Markov process whose state space consists of all sequences $(A(i))_{i\geq 1}$ where $A(i) \subset S$ are disjoint unions of balls of radius δ with the additional property that for some n we get $A(i) \neq \emptyset$ for all $i \leq n$ and $A(i) = \emptyset$ for all $i > n$.

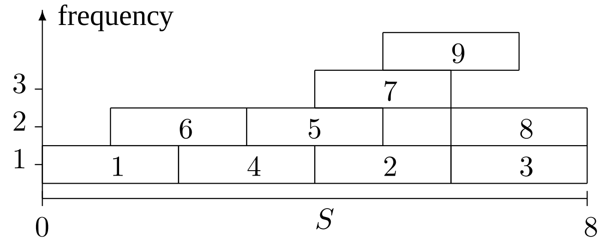

Figure 1: Assignment of frequencies to customers demanding frequencies at the following locations: 1, 5, 7, 3, 4, 2, 5, 7, 6.

Example: For illustration, we consider a one-dimensional example, where $S = [0, 8]$ and $\delta = 1$, i.e. locations of customers have to be two units apart in order to avoid interference. Assume that the first 9 customers arrive in the following order

$$1, 5, 7, 3, 4, 2, 5, 7, 6$$

Then we get the following assignment of frequencies

$$\begin{array}{lll} \Phi_1 = 1 & \Phi_2 = 1 & \Phi_3 = 1 \\ \Phi_4 = 1 & \Phi_5 = 2 & \Phi_6 = 2 \\ \Phi_7 = 3 & \Phi_8 = 2 & \Phi_9 = 4 \end{array}$$

see Figure 4 for a graphical representation of this assignment.

In what follows, we will often consider the location-frequency pair process (X_k, Φ_k), $k \geq 1$. Note that the pair process is no longer an i.i.d. process, as $\Phi_k = g_k(X_1, \ldots, X_k)$.

We are interested in the number of frequencies required to accommodate a given list of customers. If the first n demands for frequencies arrive from the locations $X_1, \ldots, X_n$, the number of different frequencies assigned is

$$F_n = \max_{1\leq k\leq n} \Phi_k = f_n(X_1, \ldots, X_n) \quad (2)$$

where $f_n : S^n \to \mathbb{N}^+$ is a function that is implicitly defined via the above considerations. We moreover introduce the doubly indexed process $(F_{m,n})_{n\in\mathbb{N},m\leq n}$

$$F_{m,n} = f_{n-m}(X_{m+1}, \ldots, X_n) \quad (3)$$

$F_{m,n}$ denotes the number of frequencies that would be required to accommodate only customers $m + 1, \ldots, n$, forgetting about the earlier customers. With this notation, we have $F_n = F_{0,n}$.

Remark: Sequential frequency assignment has some very counterintuitive properties. Two of these properties are the lack of monotonicity and the lack of

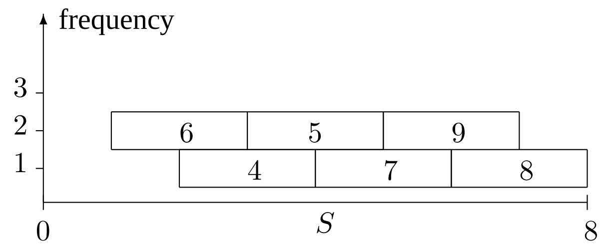

Figure 2: Assignment of frequencies to customers demanding frequencies at the following locations: 3, 4, 2, 5, 7, 6.

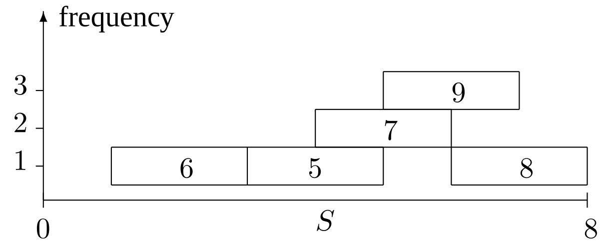

Figure 3: Assignment of frequencies to customers demanding frequencies at the following locations: 4, 2, 5, 7, 6.

subadditivity. Consider the customers in the previous example, and restrict attention to the last 6 customers, i.e. the ones with locations $3, 4, 2, 5, 7, 6$. These customers, arriving in this given order, require two frequencies. (see Figure 2) However, if we take away the first of these customers, and only consider the customers with the remaining locations $4, 2, 5, 7, 6$, we suddenly need three frequencies! (see Figure 3)

The same sequence of customers also provides an example showing that $F_{m,n}$ is not a subadditive process, i.e. that the assumption

$$F_{l,n} \leq F_{l,m} + F_{m,n}, \; \forall l, m, n \in \mathbb{N}^+, l \leq m \leq n \quad (4)$$

is generally violated. In our example, we have $F_{0,9} = 4$, $F_{0,3} = 1$, $F_{3,9} = 2$. Moreover, it is possible to find an order of locations so that the removal of one customer results in an arbitrarily large decrease of number of necessary frequencies. (see Figure 4) The reader is invited to check that by extending this example, it is always possible to construct a tower that decreases in height of order n, when only one customer is left away.

3 Conjectures and Simulations

3.1 Law of large numbers

We are in the first place interested in the asymptotic growth of F_n, the number of frequencies required to

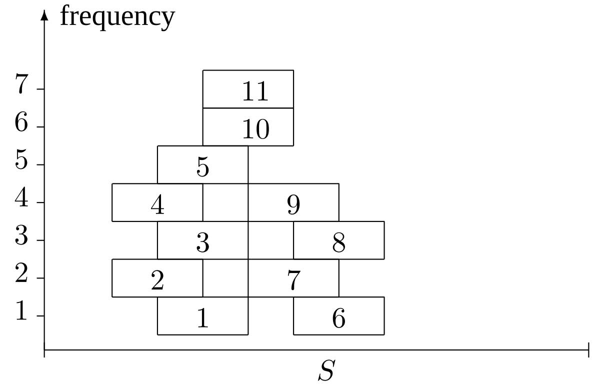

Figure 4: Possible configuration of 11 customers that arrived in such an order that if the first customer had not shown up, only 4 frequencies were necessary instead of 7.

accommodate the first n customers. Simulations support the following

Conjecture 1 *There exists a constant γ, depending on S, on δ and on the distribution of the locations X_k such that*

$$\lim_{n\to\infty} \frac{1}{n} F_n = \gamma \quad (5)$$

In the long run, sequential frequency assignment requires on average γ frequencies per customer.

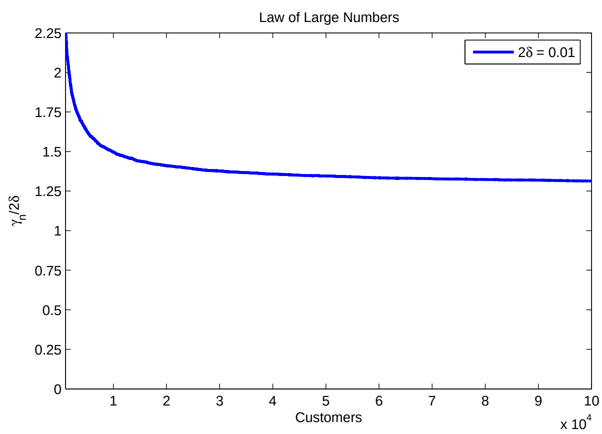

Figure 5: Simulation results suggest that $\lim_{n\to\infty} \frac{1}{n} F_n$ converges. In this figure we show the development of $\gamma_n := \frac{1}{n} F_n / 2\delta$, with $2\delta = 0.01$ in the range $n \in [10^3, 10^5]$. Here, as in all our simulations we use $S = [0, 1]$. In this particular case we find $\gamma_{10^5}/2\delta \approx 1.31$.

In the special case when $S = [0, 1]$, and the X_k are uniformly distributed over $[0, 1]$, we can study the scaling behavior of $\gamma = \gamma(\delta)$ as $\delta \to 0$. For $\delta \ll 1$, we can accommodate $(2\delta)^{-1}$ customers per frequency in the case of an ideal packing, i.e. each customer requires 2δ frequency and thus it makes sense to compare $\gamma(\delta)$ to this standard. We conjecture that there exists $\alpha \in (0,1)$ such that

$$\lim_{\delta \to 0} \frac{\gamma(\delta)}{2\delta} = \alpha^{-1} \qquad (6)$$

Both conjectures are supported by extensive simulations, see Figure 5 for an example of 1000 simulations of $\frac{1}{n}F_n/2\delta$ for $\delta = 0.01$.

At this point we do not have a mathematical proof of the law of large numbers. We believe that a proof based on Derriennic's [3] ergodic theorem for almost subadditive processes should be feasible. Though the SFAP is not subadditive, deviations from subadditivity occur very rarely. This conjecture is supported by simulations.

3.2 Central Limit Theorem

Conjecture 2 *There exists a constant* $\sigma^2 > 0$ *such that as* $n \to \infty$

$$\frac{F_n - n\gamma}{\sqrt{n}} \longrightarrow N(0, \sigma^2) \qquad (7)$$

in distribution, where $N(0,\sigma^2)$ *denotes the normal distribution with mean zero and variance* σ^2.

This conjecture is also well supported by simulations (see Figure 6).

3.3 The Active Range of Frequencies

The highest frequency in use at time n is F_n. Simulations show that the active range of frequencies, i.e. those frequencies that are in use and still have open space for new customers, does not converge but is of order $n^{\frac{1}{2}}$. In Figure 7 we picture some simulation results of the (square of the) active range as a function of the number of customers in the system.

4 Packing Density

As the number of customers increases, the frequencies become saturated one by one, i.e. for k large enough we have

$$A_k(i) \cap B_\delta(x) \neq \emptyset$$

for all $x \in S$. In other words, this frequency has no room for additional customers. We can thus define for each frequency the limit sets

$$A(i) := \bigcup_{k \geq 1} A_k(i)$$

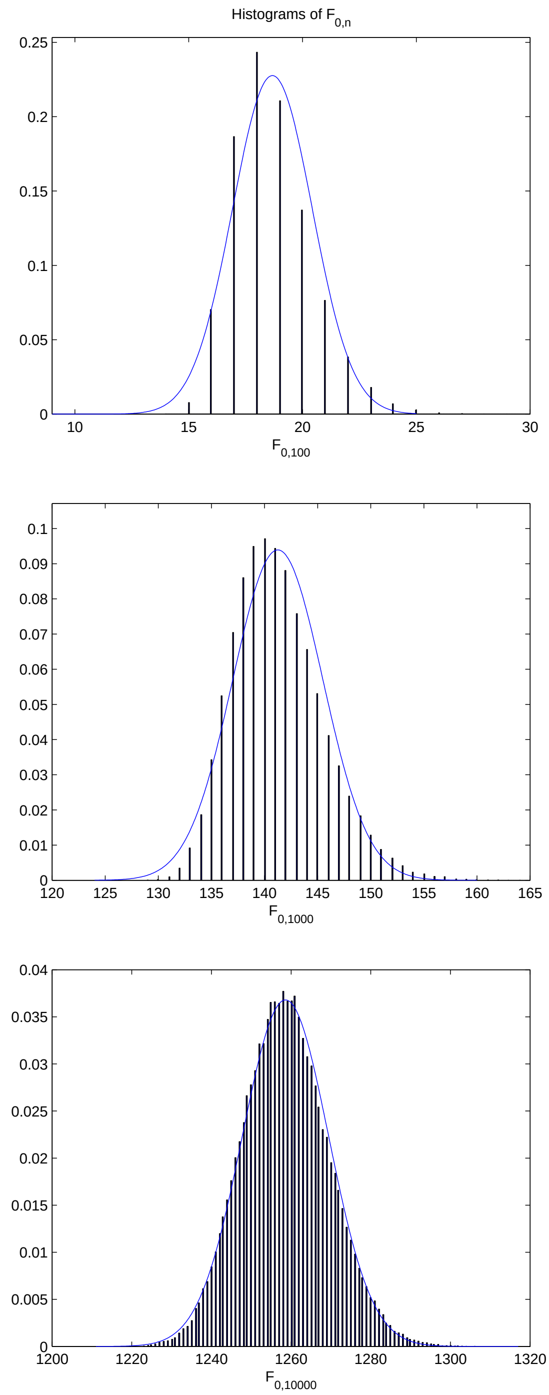

Figure 6: Histograms with results of 100,000 Monte Carlo simulations of $F_{0,100}$, $F_{0,1000}$ and $F_{0,10000}$ with $2\delta = 0.1$. The Normal probability density function with mean and variance estimated from the data is depicted in the figure.

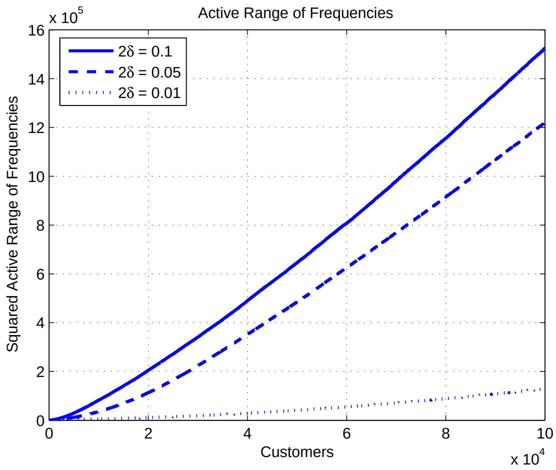

Figure 7: The active range of frequencies is of order $\sqrt{n}$. In the figure the square of the active range is plotted in the case $2\delta = 0.1$, $2\delta = 0.05$ and $2\delta = 0.01$, based on 1000 simulations.

and investigate their structure. An interesting quantity is the packing density $|A(i)|/|S|$.

In the special case when $S = [0, 1]$ and when $\delta \ll 1$, the packing density is related to the famous Renyi parking constant. In Renyi's car parking problem, first described by Renyi [4], cars of length 2δ are sequentially parked in the interval $[0, 1]$. The midpoints of the cars are uniformly distributed over $[0, 1]$. Cars are not allowed to overlap. If there is no space for a certain car, the car is rejected. This process is stopped when there is no space left for a new car. Renyi showed that the coverage of cars in the parking space converges as $\delta \to 0$ to a constant c_1. Renyi obtained an analytic formula for c_1 and showed that $c_1 \approx 0.7475$.

The process of assigning frequencies to customers can be regarded as a multilayer extension of the car parking process. In the car parking problem there is only one parking space (frequency), and cars (customers) are simply rejected when there is no space. In the SFAP model higher indexed frequencies are tested until some space is found. The first frequency is thus identical to the Renyi car parking problem. Indeed, in simulations of one-dimensional SFAP with small area customers ($\delta \ll 1$), we find that the packing density equals Renyi's parking constant.

Surprisingly, our simulations show that the packing density in the higher indexed frequencies is different from c_1 and is in fact an increasing function of the index, see Figure 8. The origin of this phenomenon is not completely understood yet. Similar phenomena have been observed in random sequential adsorption in physical chemistry (see [5], for a review).

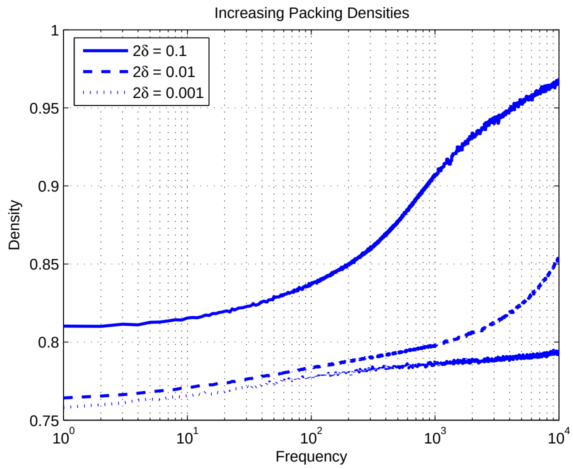

Figure 8: The packing density is an increasing function of the frequency index.

5 Conclusion

In this article we introduced a stochastic model for the one by one assignment of frequencies to customers. We investigated some of its properties and received interesting results. It is shown by counterexamples that the assignment process is not subadditive. But simulations results suggest that the mean number of needed frequencies converges to a constant value. The number of needed frequencies also tends to be normal distributed when a large number of customers is involved. Furthermore, we found that the number of active frequencies is of order $n^{\frac{1}{2}}$ when there are n customers in the system. Finally, we found that our model is in fact a multilayer extension of the well known car parking model. Some random sequential adsorption models used in physical chemistry are also related to the SFAP. Like in some of these models we find increasing densities for higher frequencies.

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