Approximation By Kantorovich Type q-Bernstein Operators

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Abstract: In the present paper, Kantorovich type of Bernstein polynomials based on q-integers is constructed. Approximation properties and rate of convergence of these operators are established with the help of the Korovkin theorem.

Key–Words: Bernstein polynomial, Kantorovich type polynomials, q-integer, Korovkin theorem, Modulus of continuity

1 Introduction

For each positive integer n, Philips [7] defined q-Bernstein polynomials as;

$$B_n(f;q,x)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1-q^s x). \tag{1}$$

When $q=1$, $B_n(f;q,x)$ is the classical Bernstein polynomial

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}. \tag{2}$$

Kantorovich [6] modified the Bernstein operators and defined the lineer positive operators $K_n : L_1([0,1]) \to C([0,1])$ defined for any $f \in L_1([0,1])$ by;

$$K_n(f;x)=(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{k/n+1}^{(k+1)/(n+1)} f(u)du, \tag{3}$$

$$p_{n,k}(x)=\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}.$$

These operators are known as Kantorovich operators in literature.

Now we recall the following definitions about q-calculus [5] .

Let $q>0$. For each nonnegative integer r, the q-integer $[r]$, q-factorial $[r]!$ and q-binomial $\begin{bmatrix} n \\ r \end{bmatrix}$, $(n \geq r \geq 0)$ are defined by

$$[r]:=[r]_q:=\begin{cases} \frac{1-q^r}{1-q} & q \neq 1, \\ r & q=1, \end{cases}$$

$$[r]!:=\begin{cases} [r][r-1]...[1] & ; \quad q \geq 1, \\ 1 & ; \quad q=1, \end{cases}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} := \frac{[n]!}{[n-r]![r]!},$$

respectively.

The q-analog of the integration in the interval $[0,b]$ is defined by [1]

$$\int_0^b f(t)d_qt=(1-q)b\sum_{j=0}^{\infty} f(q^j b)q^j \quad 0<q<1. \tag{4}$$

Note that

$$\lim_{q\to 1}\int_0^b f(x)d_qx=\int_0^b f(x)dx,$$

provided that $f(x)$ is continuous in the interval $[0,b]$.

In this paper, we will establish the q-analogue of the Bernstein-Kantorovich operators and we will examine the approximation properties of the constructed operator.

2 Approximation Properties

In this section we define the Kantorovich type q-Bernstein polynomial as;

$$B_n^*(f;q,x)=[n+1]\sum_{k=0}^{n} q^{-k}\begin{bmatrix} n \\ k \end{bmatrix} x^k \times \prod_{s=0}^{n-k-1}(1-q^s x)\int_{[k]/[n+1]}^{[k+1]/[n+1]} f(t)d_qt. \tag{5}$$

We will discuss the approximation properties of the operator (5) when we replace q in (5) by a sequence (q_n) in the interval $(0,1)$ such that

$$\lim_{n\to\infty} q_n = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{[n]} = 0 \tag{6}$$

are satisfied.

Theorem 1 *If the sequence (q_n) satisfies the conditions (6) in the interval $(0,1)$ then the operator (5) satisfy*

$$\|B_n^*(f;q) - f\| \to 0 \tag{7}$$

for every $f \in C[0,a], 0 < a < 1$.

Proof: Let us compute $B_n^*(t^s;q,x)$ for $s = 0,1,2$. We start with $s = 0$.

$$B_n^*(1;q,x) = [n+1]\sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} d_q t. \tag{8}$$

From the definition of q-integral we can write;

$$\begin{aligned}
\int_{[k]/[n+1]}^{[k+1]/[n+1]} d_q t &= \int_0^{[k+1]/[n+1]} d_q t - \int_0^{[k]/[n+1]} d_q t \\
&= (1-q)\frac{[k+1]}{[n+1]}\sum_{j=0}^{\infty} q^j \\
&\quad - (1-q)\frac{[k]}{[n+1]}\sum_{j=0}^{\infty} q^j \\
&= \frac{1-q}{[n+1]}([k+1]-[k])\sum_{j=0}^{\infty} q^j \\
&= \frac{q^k}{[n+1]}
\end{aligned}$$

where we have used the properties

$$[k+1]-[k] = q^k,$$

$$\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}, \quad \text{for} \quad 0<q<1.$$

Therefore it is easily seen from (8) that

$$B_n^*(1;q,x) = 1. \tag{9}$$

Now we will estimate $B_n^*(t^s;q,x)$ for $s = 1$. From (5) we can write,

$$B_n^*(t;q,x) = [n+1]\sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} t d_q t \tag{10}$$

Let us examine the q-integral on the right hand side of the equality.

$$\begin{aligned}
\int_{[k]/[n+1]}^{[k+1]/[n+1]} t d_q t &= \int_0^{[k+1]/[n+1]} t d_q t - \int_0^{[k]/[n+1]} t d_q t \\
&= (1-q)\frac{[k+1]}{[n+1]}\sum_{j=0}^{\infty} q^{2j}\frac{[k+1]}{[n+1]} \\
&\quad - (1-q)\frac{[k]}{[n+1]}\sum_{j=0}^{\infty} q^{2j}\frac{[k]}{[n+1]} \\
&= (1-q)\frac{[k+1]^2}{[n+1]^2}\frac{1}{1-q^2} \\
&\quad - (1-q)\frac{[k]^2}{[n+1]^2}\frac{1}{1-q^2} \\
&= \frac{1}{1+q}\frac{1}{[n+1]^2}([k+1]^2-[k]^2) \\
&= \frac{q^k}{1+q}\frac{1}{[n+1]^2}([k](1+q)+1)
\end{aligned} \tag{11}$$

Substituting (11) into (10) we get;

$$\begin{aligned}
B_n^*(t;q,x) &= [n+1]\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \\
&\quad \times \left\{\frac{1}{1+q}\frac{1}{[n+1]^2}([k](1+q)+1)\right\} \\
&= [n+1]\sum_{k=1}^{n}\left[\begin{array}{c} n \\ k \end{array}\right] x^k \frac{[k]}{[n+1]^2} \\
&\quad \times \prod_{s=0}^{n-k-1}(1-q^s x) \\
&\quad + [n+1]\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right] x^k \\
&\quad \times \prod_{s=0}^{n-k-1}(1-q^s x)\frac{1}{1+q}\frac{1}{[n+1]^2} \\
&= \frac{1}{[n+1]}\sum_{k=1}^{n}\frac{[n]!}{[n-k]![k-1]!}x^k \\
&\quad \times \prod_{s=0}^{n-k-1}(1-q^s x) + \frac{1}{1+q}\frac{1}{[n+1]}.
\end{aligned}$$

Consequently we have,

$$B_n^*(t;q,x) = \frac{[n]}{[n+1]}x + \frac{1}{1+q}\frac{1}{[n+1]} \tag{12}$$

Lastly we will examine $B_n^*(t^s;q,x)$ for $s = 2$. We have

$$B_n^*(t^2;q,x) = [n+1]\sum_{k=0}^{n} q^{-k}\left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} t^2 d_q t \qquad (13)$$

Making similar computations as in the previous cases for the q-integral, one can find

$$\begin{aligned}\int_{[k]/[n+1]}^{[k+1]/[n+1]} t^2 d_q t &= \int_0^{[k+1]/[n+1]} t^2 d_q t - \int_0^{[k]/[n+1]} t^2 d_q t \\ &= \frac{1}{[n+1]^3} \times \frac{1}{1+q+q^2} \times \left\{ q^k([k+1]^2 + [k][k+1] + [k]^2) \right\}\end{aligned} \qquad (14)$$

Substituting (14) into (13) and using the property

$$[k+1] = q[k] + 1 \qquad (15)$$

we can write,

$$\begin{aligned}B_n^*(t^2;q,x) = \frac{1}{[n+1]^2} \times \frac{1}{1+q+q^2} &\times \Bigg\{ \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times (q[k]+1)^2 \\ &+ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times (q^2[k-1]+q+1) \\ &+ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times (q[k-1]+1) \Bigg\}.\end{aligned}$$

Writing the terms explicitly, the right hand side of the equality becomes,

$$\begin{aligned}= \frac{1}{[n+1]^2}\frac{1}{1+q+q^2} &\times \Bigg\{ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) q^2 [k] \\ &+ 2\sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) q \\ &+ \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \\ &+ \sum_{k=2}^{n} \frac{[n]!}{[n-k]![k-2]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) q^2 \\ &+ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) q \\ &+ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \\ &+ \sum_{k=2}^{n} \frac{[n]!}{[n-k]![k-2]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) q \\ &+ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \Bigg\}\end{aligned}$$

Using the property given in (15) once more and then rearranging the terms yields;

$$\begin{aligned}= \frac{1}{[n+1]^2}\frac{1}{1+q+q^2} &\times \Bigg\{ \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \times q^2(q[k-1]+1) \\ &+ 2q\sum_{k=1}^{n} \frac{[n]!}{[n-k]![k-1]!} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \\ &+ 1 + [n][n-1]x^2 \sum_{k=0}^{n-2}\left[\begin{array}{c} n-2 \\ k \end{array}\right] x^k \times \prod_{s=0}^{n-k-3}(1-q^s x) q^2 \\ &+ [n]x\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-2}(1-q^s x) q \\ &+ [n]x\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-2}(1-q^s x)\end{aligned}$$

$$+ [n][n-1]x^2 \sum_{k=0}^{n-2} \left[\begin{array}{c} n-2 \\ k \end{array} \right] x^k \prod_{s=0}^{n-k-3} (1-q^s x) q$$
$$\left. + x \sum_{k=0}^{n-1} \left[\begin{array}{c} n-1 \\ k \end{array} \right] x^k \prod_{s=0}^{n-k-2} (1-q^s x) \right\}.$$

Making the necessary computations we finally get,

$$\begin{aligned} B_n^*(t^2;q,x) &= \frac{1}{[n+1]^2} \times \frac{1}{1+q+q^2} \\ &\times \left\{ [n][n-1]x^2q^3 + [n]xq^2 + 2q[n]x \right. \\ &\quad + 1 + [n][n-1]x^2q^2 + [n]xq + [n]x \\ &\quad \left. + [n][n-1]x^2q + [n]x \right\}. \end{aligned}$$

More clearly,we have,

$$\begin{aligned} B_n^*(t^2;q,x) &= \frac{[n][n-1]}{[n+1]^2} \frac{q^3+q^2+q}{1+q+q^2} x^2 \\ &+ \frac{[n]}{[n+1]^2} \frac{q^2+3q+2}{1+q+q^2} x \\ &+ \frac{1}{[n+1]^2} \frac{1}{1+q+q^2}. \end{aligned} \tag{16}$$

Consequently replacing q by a sequence (q_n) such that $\lim\limits_{n\to\infty} q_n = 1$ and taking the property (6) into account, from (9), (12) and (16) we can write,

$$\begin{aligned} B_n^*(1;q_n,x) &\rightrightarrows 1 \\ B_n^*(t;q_n,x) &\rightrightarrows x \\ B_n^*(t^2;q_n,x) &\rightrightarrows x^2, \end{aligned}$$

respectively. Therefore the conditions of the Korovkin's theorem are satisfied and the proof of the theorem is completed.

Remark 1: For the special case $q = 1$ we have;

$$\begin{aligned} B_n^*(1;x) &= 1 \\ B_n^*(t;x) &= \frac{n}{n+1}x + \frac{1}{2(n+1)} \\ B_n^*(t^2;x) &= \frac{n(n-1)}{(n+1)^2}x^2 + \frac{2n}{(n+1)^2}x + \frac{1}{3(n+1)^2} \end{aligned}$$

Remark 2: The first and the second moment of the operator $B_n^*(f;q,x)$ are

$$B_n^*((s-x);q,x) = \left(\frac{[n]}{[n+1]} - 1 \right) x + \frac{1}{[n+1]} \frac{1}{1+q}$$

and

$$\begin{aligned} B_n^*((s-x)^2;q,x) &= \frac{[n][n-1]}{[n+1]^2} \left\{ \frac{q^3+q^2+q}{q^2+q+1} \right\} x^2 \\ &+ \frac{[n]}{[n+1]^2} \left\{ \frac{q^2+3q+2}{q^2+q+1} \right\} x \\ &+ \frac{1}{[n+1]^2} \frac{q}{q^2+q+1} \\ &+ x^2 - 2\frac{[n]}{[n+1]}x^2 - 2\frac{1}{1+q}\frac{1}{[n+1]}x, \end{aligned}$$

respectively.

3 Order of Approximation

In this section, we compute the approximation order of the operator $B_n^*(f;q,x)$ by means of modulus of continuity.

Let $f \in C[0,a]$. The modulus of continuity of f, $w(f,\delta)$, is defined by

$$w(f;\delta) = \sup_{\substack{|x-y|\le\delta \\ x,y\in[0,a]}} |f(x)-f(y)| \tag{17}$$

It is well-known that, for a function $f \in C[0,a]$ we have

$$\lim_{\delta\to0^+} w(f;\delta) = 0 \tag{18}$$

and

$$|f(x)-f(y)| \le w(f;\delta)\left(\frac{|x-y|}{\delta} + 1 \right). \tag{19}$$

for any $\delta > 0$.

The following theorem gives the rate of convergence of the sequence $B_n^*(f;q,x)$ by means of modulus of continuity.

Theorem 2 *If the sequence $q := (q_n)$ satisfies the conditions given in (6), then*

$$\|B_n^*(f;q) - f\| \le 2w(f,\sqrt{\delta_n}) \tag{20}$$

for all $f \in C[0,a]$,where

$$\delta_n = B_n^*((s-x)^2;q;x). \tag{21}$$

Proof: Let $f \in C[0,a]$. From the linearity and monotonicity of $B_n^*(f;q,x)$ we can write,

$$\begin{aligned} &|B_n^*(f;q,x) - f(x)| \\ &\le B_n^*(|f(t)-f(x)|;q,x) \\ &= [n+1]\sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array} \right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \\ &\quad \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} |f(t)-f(x)| d_q t \end{aligned} \tag{22}$$

On the other hand

$$|f(t) - f(x)| \leq w(f; |t - x|).$$

If $|t - x| < \delta$, it is obvious that

$$|f(t) - f(x)| \leq \left(1 + \frac{(t-x)^2}{\delta^2}\right) w(f, \delta). \qquad (23)$$

If $|t - x| > \delta$, we use the following property

$$w(f, \lambda\delta) \leq (1 + \lambda) w(f, \delta) \leq (1 + \lambda^2) w(f, \delta),$$

where we choose $\lambda \in R^+$ as $\lambda = \frac{|t-x|}{\delta}$. Therefore we have,

$$|f(t) - f(x)| \leq \left(1 + \frac{(t-x)^2}{\delta^2}\right) w(f, \delta) \qquad (24)$$

for $|t-x| > \delta$. Consequently by means (23) and (24), from (22) we get,

$$\begin{aligned} &|B_n^*(f; q, x) - f(x)| \\ &\leq [n+1] \sum_{k=0}^{n} q^{-k} \left[\begin{array}{c} n \\ k \end{array} \right] x^k \prod_{s=0}^{n-k-1} (1 - q^s x) \\ &\quad \int_{[k]/[n+1]}^{[k+1]/[n+1]} \left(1 + \frac{(s-x)^2}{\delta^2}\right) w(f, \delta) d_q t \\ &= \left\{ B_n^*(1; q, x) + \frac{1}{\delta^2} B_n^*((s-x)^2; q, x) \right\} w(f; \delta) \\ &= \left\{ 1 + \frac{1}{\delta^2} B_n^*((s-x)^2; q, x) \right\} w(f; \delta). \qquad (25) \end{aligned}$$

Taking (6) and Remark 2 into account one can easily obtain that

$$\lim_{n \to \infty} B_n^*((s-x)^2; q_n, x) = 0.$$

So letting $\delta_n = B_n^*((s-x)^2; q_n, x)$ and taking $\delta = \sqrt{\delta_n}$, we finally get

$$\|B_n^*(f; q, x) - f(x)\| \leq 2w(f; \sqrt{\delta_n}), \qquad (26)$$

as desired.

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