Approximation Properties of Bivariate Generalization of Bleimann, Butzer and Hahn Operators Based on the q-Integers

SİBEL ERSAN
Department of Mathematics and Computer Sciences
Cankaya University, Faculty of Arts and Sciences
06530, Balgat Ankara
TURKEY

Abstract: In this presentation, bivariate case of BBH operators based on the q- integers is constructed. Then Korovkin type approximation properties of this generalization are obtained with the help of Volkov's Theorem. Lastly, we obtain rates of convergence of these operators by means of bivariate modulus of continuity and Lipschitz type maximal functions.

Key–Words: Positive linear operators, bivariate Korovkin theorem, bivariate modulus of continuity, bivariate Lipschitz type maximal function, q-integers.

1 Introduction

In [1] , Bleimann, Butzer and Hahn (BBH) introduced the following operator, that is for $x \geq 0$;

$$L_n(f;x) = (1+x)^{-n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\left(\begin{array}{c} n \\ k \end{array}\right) x^k, \tag{1}$$

They investigated pointwise convergence properties of (1) in a compact sub-interval of $[0, \infty)$.
Then Gadjiev and Çakar [2] obtained uniform convergence of (1) on semi-axis $[0, \infty)$ on some subspace of bounded and continuous functions by using the test functions $(\frac{x}{1+x})^{\nu}, \nu = 0, 1, 2$.
In 1996 q-based generalization of the classical Bernstein polynomials were introduced by G. M. Phillips [3]. He has obtained rate of convergence for the Bernstein polynomials based on q-integers. Firstly let us give some definitions about q-integers [5]:
For any fixed real number $q > 0$ and non-negative integer k, the q-integer of the number k is defined by

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1 \end{cases}$$

The q-factorial is defined in the following:

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \dots [1]_q, & k = 1, 2, .. \\ 1, & k = 0 \end{cases}$$

and q-binomial coefficient is defined as

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} \qquad (n \geq k \geq 0)$$

Recently Aral and Doğru [4] gave a new generalization of BBH operators based on q-integers as follows: For $x \geq 0,\ f : R_+ \to R$ and $0 < q \leq 1$

$$L_n(f;q,x) = \frac{1}{l_{n,q}(x)}\sum_{k=0}^{n} f\left(\frac{[k]_q}{[n-k+1]_q\, q^k}\right) q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k,$$

where

$$l_{n,q}(x) = \prod_{s=0}^{n-1} (1 + q^s x)\,.$$

They investigate uniform approximation of these operators on some subspace of bounded and continuous functions.
The bivariate case for the operators are first introduced by D.D. Stancu [6]. He studied the bivariate Bernstein polynomials and estimated the order of approximation for these operators.
The aim of this paper is to construct bivariate q-BBH operators, investigate Korovkin type approximation properties and estimate the order of approximation in terms of a modulus of continuity.

2 Construction of the Bivariate Operators

Let $R_+^2 = [0,\infty) \times [0,\infty)$, $f : R_+^2 \to R$ and $0 < q_{n_1}, q_{n_2} \le 1$. We define the bivariate extension of the q-Bleimann-Butzer ve Hahn operators as follows:

$$L_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) = \frac{1}{l_{n_1,q_{n_1}}(x)}\frac{1}{l_{n_2,q_{n_2}}(y)} \times \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{[k_1]}{[n_1-k_1+1]\,q_{n_1}^{k_1}}, \frac{[k_2]}{[n_2-k_2+1]\,q_{n_2}^{k_2}}\right) \times q_{n_1}^{\frac{k_1(k_1-1)}{2}} q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} x^{k_1} y^{k_2} \tag{2}$$

Here $l_{n_1,q_{n_1}}(x) = \prod_{s=0}^{n_1-1}\left(1+q_{n_1}^s x\right)$.

It is easy to check that (2) is linear and positive. By choosing $q_{n_1} = q_{n_2} = 1$, BBH operators reduce to the classical bivariate BBH operators given by

$$L_{n_1,n_2}(f;x,y) = \frac{1}{(1+x)^{n_1}}\frac{1}{(1+y)^{n_2}} \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\frac{k_1}{n_1-k_1+1}, \frac{k_2}{n_2-k_2+1}\right)\binom{n_1}{k_1}\binom{n_2}{k_2} x^{k_1}y^{k_2}.$$

Now let us give some lemmas which are often used.

Lemma 1 *The operator (2) satisfies these conditions:*

1. $L_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) = A_{n_1}^x(B_{n_2}^y(f;q_{n_2},x,y))$,
2. $L_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) = B_{n_2}^y(A_{n_1}^x(f;q_{n_1},x,y))$.

Here

$$A_{n_1}^x(f;q_{n_1},x,y) = \frac{1}{l_{n_1,q_{n_1}}(x)} \times \sum_{k_1=0}^{n_1} f\left(\frac{[k_1]}{[n_1-k_1+1]\,q_{n_1}^{k_1}}, y\right) \times q_{n_1}^{\frac{k_1(k_1-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}} x^{k_1},$$

$$B_{n_2}^y(f;q_{n_2},x,y) = \frac{1}{l_{n_2,q_{n_2}}(y)} \times \sum_{k_2=0}^{n_2} f\left(x, \frac{[k_2]}{[n_2-k_2+1]\,q_{n_2}^{k_2}}\right) \times q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} y^{k_2}$$

Proof: 1. $A_{n_1}^x(B_{n_2}^y(f;q_{n_2},x,y))$

$$= A_{n_1}^x\left(\frac{1}{l_{n_2,q_{n_2}}(y)}\sum_{k_2=0}^{n_2} f(x, \tfrac{[k_2]}{[n_2-k_2+1]q_{n_2}^{k_2}})\, q_{n_2}^{\frac{k_2(k_2-1)}{2}} \times \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} y^{k_2}\right)$$

$$= \frac{1}{l_{n_2,q_{n_2}}(y)}\sum_{k_2=0}^{n_2} A_{n_1}^x\left(f(x, \tfrac{[k_2]}{[n_2-k_2+1]q_2^{k_2}}), q_{n_1}, x, y\right) \times q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} y^{k_2}$$

$$= \frac{1}{l_{n_2,q_{n_2}}(y)}\sum_{k_2=0}^{n_2} q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} y^{k_2} \sum_{k_1=0}^{n_1}\frac{1}{l_{n_1,q_{n_1}}(x)} \times f(\tfrac{[k_1]}{[n_1-k_1+1]q_{n_1}^{k_1}}, \tfrac{[k_2]}{[n_2-k_2+1]q_{n_2}^{k_2}}) q_{n_1}^{\frac{k_1(k_1-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}} x^{k_1}$$

$$= \frac{1}{l_{n_1,q_{n_1}}(x)}\frac{1}{l_{n_2,q_{n_2}}(y)}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} f\left(\tfrac{[k_1]}{[n_1-k_1+1]q_{n_1}^{k_1}}, \times \tfrac{[k_2]}{[n_2-k_2+1]q_{n_2}^{k_2}}\right) q_{n_1}^{\frac{k_1(k_1-1)}{2}} q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}} \times \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} x^{k_1}y^{k_2}$$

$= A_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$
2. can be proven in a similar manner.

Lemma 2 *Let $\tilde{e_{ij}} : R_+^2 \to [0,1)$ be the two dimensional test function defined as $\tilde{e_{ij}} = (\frac{x}{1+x})^i(\frac{y}{1+y})^j$. Then we have the following items for the operator (2):*

$$i) L_{n_1,n_2}(\tilde{e_{00}};q_{n_1},q_{n_2},x,y) = 1,$$

$$ii) L_{n_1,n_2}(\tilde{e}_{10};q_{n_1},q_{n_2},x,y) = \frac{[n_1]}{[n_1+1]}\frac{x}{1+x},$$

$$iii) L_{n_1,n_2}(\tilde{e}_{01};q_{n_1},q_{n_2},x,y) = \frac{[n_2]}{[n_2+1]}\frac{y}{1+y},$$

$$iv) L_{n_1,n_2}(\tilde{e}_{20};q_{n_1},q_{n_2},x,y) = \frac{[n_1][n_1-1]}{[n_1+1]^2}q_{n_1}^2 \times \frac{x^2}{(1+x)(1+q_{n_1}x)} + \frac{[n_1]}{[n_1+1]^2}\frac{x}{1+x},$$

$$v) L_{n_1,n_2}(\tilde{e}_{02};q_{n_1},q_{n_2},x,y) = \frac{[n_2][n_2-1]}{[n_2+1]^2}q_{n_2}^2 \times \frac{y^2}{(1+y)(1+q_{n_2}y)} + \frac{[n_2]}{[n_2+1]^2}\frac{y}{1+y}.$$

Proof:

$$i) L_{n_1,n_2}(\tilde{e}_{00};q_{n_1},q_{n_2},x,y) = \frac{1}{l_{n_1,q_1}(x)}\frac{1}{l_{n_2,q_2}(y)} \times \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} q_{n_1}^{\frac{k_1(k_1-1)}{2}} q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}}$$

For $0 \le q \le 1$

$$\sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=0}^{n-1}(1+q^k x) = l_{n,q}(x).$$

So (i) holds [5].

$$ii) \quad L_{n_1,n_2}(\tilde{e}_{10}; q_{n_1}, q_{n_2}, x, y) = \frac{1}{l_{n_1,q_{n_1}}(x)} \frac{1}{l_{n_2,q_{n_2}}(y)}$$
$$\times \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \frac{[k_1]}{[n_1+1]} q_{n_1}^{\frac{k_1(k_1-1)}{2}} q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}}$$
$$\times \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} x^{k_1} y^{k_2}$$
$$= \frac{1}{l_{n_1,q_{n_1}}(x)} \frac{[n_1]}{[n_1+1]} \sum_{k_1=1}^{n_1} q_{n_1}^{\frac{k_1(k_1-1)}{2}} \begin{bmatrix} n_1-1 \\ k_1-1 \end{bmatrix}_{q_{n_1}}$$
$$\times x^{k_1}$$
$$= \frac{x}{l_{n_1,q_{n_1}}(x)} \frac{[n_1]}{[n_1+1]} \sum_{k_1=0}^{n_1-1} q_{n_1}^{\frac{k_1(k_1-1)}{2}} \begin{bmatrix} n_1-1 \\ k_1 \end{bmatrix}_{q_{n_1}}$$
$$\times (q_{n_1} x)^{k_1}$$
$$= \frac{[n_1]}{[n_1+1]} \frac{x}{1+x}.$$

iii) can be proven in a similar way.
iv) With a direct computation we have

$$L_{n_1,n_2}(\tilde{e}_{20}; q_{n_1}, q_{n_2}, x, y) = \frac{1}{l_{n_1,q_{n_1}}(x)} \frac{1}{l_{n_2,q_{n_2}}(y)}$$
$$\times \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \frac{[k_1]^2}{[n_1+1]^2} q_{n_1}^{\frac{k_1(k_1-1)}{2}} q_{n_2}^{\frac{k_2(k_2-1)}{2}} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_{n_1}}$$
$$\times \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_{n_2}} x^{k_1} y^{k_2}$$
$$= \frac{1}{l_{n_1,q_{n_1}}(x)} \frac{[n_1-1][n_1]}{[n_1+1]^2} \sum_{k_1=2}^{n_1} [k_1-1]\, q_{n_1}$$
$$\times \begin{bmatrix} n_1-2 \\ k_1-2 \end{bmatrix}_{q_{n_1}} q_{n_1}^{\frac{k_1(k_1-1)}{2}} x^{k_1} + \frac{1}{l_{n_1,q_{n_1}}(x)}$$
$$\frac{[n_1]}{[n_1+1]^2} \sum_{k_1=1}^{n_1} \begin{bmatrix} n_1-1 \\ k_1-1 \end{bmatrix}_{q_{n_1}} q_{n_1}^{\frac{k_1(k_1-1)}{2}} x^{k_1}$$
$$= \frac{x^2}{l_{n_1,q_{n_1}}(x)} q_{n_1}^2 \frac{[n_1-1][n_1]}{[n_1+1]^2} \sum_{k_1=0}^{n_1-2} \begin{bmatrix} n_1-2 \\ k_1 \end{bmatrix}_{q_{n_1}}$$
$$q_{n_1}^{\frac{k_1(k_1-1)}{2}} (q_{n_1}^2 x)^{k_1} + \frac{x}{l_{n_1,q_{n_1}}(x)} \frac{[n_1]}{[n_1+1]^2} \sum_{k_1=0}^{n_1-1}$$
$$\begin{bmatrix} n_1-1 \\ k_1 \end{bmatrix}_{q_{n_1}} q_{n_1}^{\frac{k_1(k_1-1)}{2}} (q_{n_1} x)^{k_1}$$
$$= \frac{[n_1][n_1-1]}{[n_1+1]^2} q_{n_1}^2 \frac{x^2}{(1+x)(1+q_{n_1}x)} + \frac{[n_1]}{[n_1+1]^2} \frac{x}{1+x}$$

v) Obvious.

3 Approximation Properties of Bivariate Operators

In this section some theorems on uniform convergence for bivariate case will be given.
Let $C_B(R_+^2)$ be the space of all bounded and continuous functions on R_+^2. Then $C_B(R_+^2)$ is a linear normed space with

$$\|f\|_{C_B(R_+^2)} = \sup_{x,y \ge 0} |f(x,y)|.$$

If

$$\lim_{n,m\to\infty} \|f_{n,m} - f\|_{C_B(R_+^2)} = 0$$

holds, then we say that the sequence $\{f_{n,m}\}$ converges uniformly to f and it's shown as $f_{n,m} \rightrightarrows f$.
Now, let us introduce modulus of continuity type function $w(\delta)$, so that the following conditions are satisfied:
i)$w(\delta)$ is nonnegative and increasing for δ,

ii)$w(\delta_1+\delta_2) \le w(\delta_1) + w(\delta_2)$,
iii) $\underset{\delta\to 0}{lim} w(\delta) = 0$.

Let H_w be the subspace of real valued functions satisfying $\forall x, y \in R_+$

$$|f(x) - f(y)| \le w\left(\left|\frac{x}{1+x} - \frac{y}{1+y}\right|\right).$$

It can be obtained that $H_w \subset C_B(R_+)$ for the bounded and continuous functions f on R_+.
For example, if we choose

$$w(t) = Mt^\alpha \ , \ 0 < \alpha \le 1$$

we have

$$|f(x) - f(y)| \le M \frac{|x-y|^\alpha}{(1+x)^\alpha (1+t)^\alpha}$$

so that it can be easily seen that $H_w \subset Lip_M \alpha$.

Theorem 3 *[5] Let A_n be the sequence of linear positive operators acting from $H_w(R_+)$ to $C_B(R_+)$satisfying*

$$\underset{n\to\infty}{lim} \left\| A_n((\frac{t}{1+t})^\nu; x) - (\frac{x}{1+x})^\nu \right\|_{C_B} = 0, \nu = 0, 1, 2$$

then for any function $f \in H_w(R_+)$

$$\underset{n\to\infty}{lim} \|A_n(f) - f\|_{C_B} = 0$$

holds.

Now let us demonstrate that Theorem 3 also holds for the bivariate case:

Theorem 4 *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ satisfies $0 < q_{n_1} \leq 1$, $0 < q_{n_2} \leq 1$ and let $q_{n_1} \to 1$ and $q_{n_2} \to 1$ for $n_1, n_2 \to \infty$. If the sequence of linear positive operator A_{n_1,n_2};*
$A_{n_1,n_2} : H_w(R_+^2) \to C_B(R_+^2)$ satisfies the following conditions;

$$i) \lim_{n_1,n_2\to\infty} \|A_{n_1,n_2}(\tilde{e_{00}}; q_{n_1}, q_{n_2}, x, y) - \tilde{e_{00}}\|_{C(R_+^2)} = 0, \quad (3)$$

$$ii) \lim_{n_1,n_2\to\infty} \|A_{n_1,n_2}(\tilde{e_{10}}; q_{n_1}, q_{n_2}, x, y) - \tilde{e_{10}}\|_{C(R_+^2)} = 0, \quad (4)$$

$$iii) \lim_{n_1,n_2\to\infty} \|A_{n_1,n_2}(\tilde{e_{01}}; q_{n_1}, q_{n_2}, x, y) - \tilde{e_{01}}\|_{C(R_+^2)} = 0, \quad (5)$$

$$iv) \lim_{n_1,n_2\to\infty} \|A_{n_1,n_2}(\tilde{e_{20}} + \tilde{e_{02}}; q_{n_1}, q_{n_2}, x, y) - (\tilde{e_{20}} + \tilde{e_{02}})\|_{C(R_+^2)} = 0, \quad (6)$$

then for any function f, $f \in H_w(R_+^2)$

$$\lim_{n_1,n_2\to\infty} \|A_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)\|_{C(R_+^2)} = 0$$

holds. Here $\tilde{e_{ij}} : R_+^2 \to [0,1)$; $\tilde{e_{ij}} = (\frac{x}{1+x})^i(\frac{y}{1+y})^j$ are two dimensional test functions. On $H_w(R_+^2)$,

$$|f(t,s) - f(x,y)| \leq w\left(\left|\left(\frac{t}{1+t}, \frac{s}{1+s}\right) - \left(\frac{x}{1+x}, \frac{y}{1+y}\right)\right|\right)$$

is defined as for bivariate case.

Proof: If $f \in H_w(R_+^2)$ then we have for any $\varepsilon > 0$ there exists a neighbourhood δ such that

$$|f(t,s) - f(x,y)| < \varepsilon$$

if $\sqrt{(\frac{t}{1+t} - \frac{x}{1+x})^2 + (\frac{s}{1+s} - \frac{y}{1+y})^2} < \delta$.
Also, boundness of f implies that there exists a positive constant M such that

$$|f(t,s) - f(x,y)| \leq \frac{2M}{\delta^2}\left[(\frac{t}{1+t} - \frac{x}{1+x})^2 + (\frac{s}{1+s} - \frac{y}{1+y})^2\right]$$

if $\sqrt{(\frac{t}{1+t} - \frac{x}{1+x})^2 + (\frac{s}{1+s} - \frac{y}{1+y})^2} \geq \delta$.
Therefore, for all $(t,s), (x,y) \in R_+^2$

$$|f(t,s) - f(x,y)| \leq \varepsilon + \frac{2M}{\delta^2}\left[(\frac{t}{1+t} - \frac{x}{1+x})^2 + (\frac{s}{1+s} - \frac{y}{1+y})^2\right].$$

holds. Applying the operator A_{n_1,n_2} to the above inequality we get

$$\begin{aligned} |A_{n_1,n_2}(f) - f|_{C(R_+^2)} &\leq (\varepsilon + M)\,|A_{n_1,n_2}(\tilde{e_{00}}) - \tilde{e_{00}}| \\ &+ \varepsilon + \frac{2M}{\delta^2}\left[|A_{n_1,n_2}(\tilde{e_{20}} + \tilde{e_{02}}) - (\tilde{e_{20}} + \tilde{e_{02}})|\right. \\ &\left.+ 2\,|A_{n_1,n_2}(\tilde{e_{10}}) - \tilde{e_{10}}| + 2\,|A_{n_1,n_2}(\tilde{e_{01}}) - \tilde{e_{01}}|\right]. \end{aligned}$$

By using the conditions (3)-(6) we find

$$\lim_{n\to\infty} \|A_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)\|_{C(R_+^2)} = 0$$

as desired. Now let us show that this theorem also holds for bivariate q-Bleimann,Butzer and Hahn operators.

Theorem 5 *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ satisfies $0 < q_{n_1} \leq 1$, $0 < q_{n_2} \leq 1$ and let $q_{n_1} \to 1$ and $q_{n_2} \to 1$ for $n_1, n_2 \to \infty$. If the sequence of linear positive operator $L_{n_1,n_2} : H_w(R_+^2) \to C_B(R_+^2)$ satisfy conditions (3)-(6), then L_{n_1,n_2} converges uniformly to f in R_+^2 for all $f \in H_w(R_+^2)$. That is; $\forall f \in H_w(R_+^2)$*

$$\lim_{n\to\infty} \|L_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)\|_{C(R_+^2)} = 0$$

Here $\tilde{e_{ij}} = (\frac{x}{1+x})^i(\frac{y}{1+y})^j$.

Proof: Below results can be obtained by using Lemma 2

$$\lim_{n_1,n_2\to\infty} \|L_{n_1,n_2}(\tilde{e_{00}}; q_{n_1}, q_{n_2}, x, y) - 1\| = 0$$

is obvious.

$$\|L_{n_1,n_2}(\tilde{e_{10}}; q_{n_1}, q_{n_2}, x, y) - \tilde{e_{10}}\| = \sup_{x,y\geq 0}\left|\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}\frac{x}{1+x} - \frac{x}{1+x}\right| \leq \left|\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}} - 1\right|$$

Since $\lim_{n\to\infty}\frac{[n]}{[n+1]} = 1$, (4) is justified. Similarly (5) can also be shown.
After simple calculation

$$\frac{[n]\,[n-1]}{[n+1]^2} = \frac{1}{q^3}\left(1 - \frac{2+q}{[n+1]} + \frac{1+q}{[n+1]^2}\right)$$

can easily be found. When this result is substituted in below equality, followings can be obtained.

$$\begin{aligned}&\left\| L_{n_1,n_2}\left(\tilde{e_{20}}+\tilde{e_{02}};q_{n_1},q_{n_2},x,y\right)-\left(\tilde{e_{20}}+\tilde{e_{02}}\right)\right\| \\ &=\sup_{x,y\geq 0}\left|\frac{[n_1]_{q_{n_1}}[n_1-1]_{q_{n_1}}}{[n_1+1]^2_{q_{n_1}}}q^2_{n_1}\frac{x^2}{(1+x)(1+q_{n_1}x)}\right. \\ &\qquad+\frac{[n_1]_{q_{n_1}}}{[n_1+1]^2_{q_{n_1}}}\frac{x}{1+x} \\ &\quad+\frac{[n_2]_{q_{n_2}}[n_2-1]_{q_{n_2}}}{[n_2+1]^2_{q_{n_2}}}q^2_{n_2}\frac{y^2}{(1+y)(1+q_{n_2}y)} \\ &\quad\left.+\frac{[n_2]}{[n_2+1]^2}\frac{y}{1+y}-\frac{x^2}{(1+x)^2}-\frac{y^2}{(1+y)^2}\right| \\ &\leq\left(\frac{1}{q^2_{n_1}}-1\right)+\frac{1}{q^2_{n_1}}\left(\frac{2+q_{n_1}}{[n_1+1]_{q_{n_1}}}-\frac{1+q_{n_1}}{[n_1+1]^2_{q_{n_1}}}\right) \\ &\quad+\frac{1}{q_{n_1}[n_1+1]_{q_{n_1}}}-\frac{1}{q_{n_1}[n_1+1]^2_{q_{n_1}}} \\ &+\left(\frac{1}{q^2_{n_2}}-1\right)+\frac{1}{q^2_{n_2}}\left(\frac{2+q_{n_2}}{[n_2+1]_{q_{n_2}}}-\frac{1+q_{n_2}}{[n_2+1]^2_{q_{n_2}}}\right) \\ &\quad+\frac{1}{q_{n_2}[n_2+1]_{q_{n_2}}}-\frac{1}{q_{n_2}[n_2+1]^2_{q_{n_2}}}.\end{aligned}$$

Since $[n+1]\to\infty$ for $n\to\infty$ and $q\to 1$

$$\lim_{n_1,n_2\to\infty}\left\| L_n(\tilde{e_{20}}+\tilde{e_{02}};q_{n_1},q_{n_2},x,y)-(\tilde{e_{20}}+\tilde{e_{02}})\right\|=0$$

is found. So we have $\forall f\in H_w(R^2_+)$

$$\lim_{n_1,n_2\to\infty}\left\| L_{n_1,n_2}(f)-f\right\|_{C_B(R^2_+)}=0$$

can be obtained by using Theorem 4.

4 Rates of convergence of the bivariate operators

In this section rate of convergence of (2) will be established by means of some bivariate modulus of smoothness.
Modulus of continuity for bivariate case is defined as follows: $f\in H_w(R^2_+)$:

$$\begin{aligned}\tilde{w}(f;\delta_1,\delta_2)&=\sup_{t,x\geq 0}\{|f(t,s)-f(x,y)|; \\ &\left|\frac{t}{1+t}-\frac{x}{1+x}\right|\leq\delta_1,\left|\frac{s}{1+s}-\frac{y}{1+y}\right|\leq\delta_2, \\ &\qquad (t,s)\in R^2_+,(x,y)\in R^2_+\}.\end{aligned}$$

Here $\tilde{w}(f;\delta_1,\delta_2)$ is satisfied following conditions $\forall f\in H_w(R^2_+)$:
i) $\tilde{w}(f;\delta_1,\delta_2)\to 0$ if $\delta_1\to 0$ and $\delta_2\to 0$,

$$\begin{aligned}&ii)\ |f(t,s)-f(x,y)|\leq \\ &\tilde{w}(f;\delta_1,\delta_2)\left(1+\frac{\left|\frac{t}{1+t}-\frac{x}{1+x}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{s}{1+s}-\frac{y}{1+y}\right|}{\delta_2}\right).\end{aligned}\tag{7}$$

Theorem 6 *Let $q=(q_{n_1})$ and $q=(q_{n_2})$ satisfies $0<q_{n_1}\leq 1$, $0<q_{n_2}\leq 1$ and $q_{n_1}\to 1$ and $q_{n_2}\to 1$ for $n_1,n_2\to\infty$. So we have*

$$\begin{aligned}&|L_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\leq \\ &\qquad 4\tilde{w}(f;\delta_1(x),\delta_2(y))\end{aligned}$$

$\forall f\in H_w(R^2_+)$ and $x,y\geq 0$. Here

$$\begin{aligned}&\delta_k(x)= \\ &\left[\frac{x^2}{(1+x)^2}\left(\frac{q_{n_k}[n_k]_{q_{n_k}}[n_k-1]_{q_{n_k}}}{[n_k+1]^2_{q_{n_k}}}\frac{(1+x)}{(1+q_{n_k}x)}\right.\right. \\ &\left.\left.-\frac{2[n_k]_{q_{n_k}}}{[n_k+1]_{q_{n_k}}}+1\right)+\frac{[n_k]_{q_{n_k}}}{[n_k+1]^2_{q_{n_k}}}\frac{x}{1+x}\right]^{\frac{1}{2}}, \\ &\qquad k=1,2.\end{aligned}\tag{8}$$

Proof: Applying the operators (2) to the inequality (7) and according to the Cauchy-Schwarz inequality, the proof is easily obtained.

In $E\times E\subset R_+\times R_+$, let Lipschitz type maximal function space is defined as follows:

$$\begin{aligned}&\tilde{W}_{\alpha_1,\alpha_2,E^2}= \\ &\{f:\sup(1+t)^{\alpha_1}(1+s)^{\alpha_2}(f_{\alpha_1,\alpha_2}(t,s)-f_{\alpha_1,\alpha_2}(x,y)) \\ &\leq M\frac{1}{(1+x)^{\alpha_1}}\frac{1}{(1+y)^{\alpha_2}};x,y\geq 0,(t,s)\in E^2\right\}.\end{aligned}$$

Here f is bounded and continuous function in R_+, M is positive constant and $0\leq\alpha_1\leq 1$, $0\leq\alpha_2\leq 1$ then f_{α_1,α_2} is the following function as;

$$f_{\alpha_1,\alpha_2}(t,s)-f_{\alpha_1,\alpha_2}(x,y)=\frac{|f(t,s)-f(x,y)|}{|t-x|^{\alpha_1}|s-y|^{\alpha_2}}.$$

In addition that $d(x,E)$ is the distance between x and E and this is also shown as $d(x,E)=\inf\{|x-y|;y\in E\}$.

Theorem 7 $0 < \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ *and* $\forall f \in W_{\alpha_1,\alpha_2,E^2}$ *we get*

$$|L_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)| \leq$$
$$M\left[(\delta_1(x))^{\alpha_1}(\delta_2(y))^{\alpha_2} + 2(d(x,E))^{\alpha_1}(d(y,E))^{\alpha_2}\right].$$

where $\delta_1(x)$ *and* $\delta_2(y)$*; defined as in (8).*

Proof: Let $x, y \geq 0$ and $(x_0, y_0) \epsilon E$. So it can be written as

$$\begin{aligned} |f(t,s) - f(x,y)| &= |f(t,s) - f(x_0,y_0) \\ &\quad + f(x_0,y_0) - f(x,y)| \\ &\leq |f(t,s) - f(x_0,y_0)| + |f(x_0,y_0) - f(x,y)| \end{aligned}$$

Applying (2) to the inequality and $\forall f \in W_{\alpha_1,\alpha_2,E^2}$, followings are obtained.

$$|L_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)| \leq$$
$$M\left(L_{n_1,n_2}\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{s}{1+s} - \frac{y_0}{1+y_0}\right|^{\alpha_2}\right)$$
$$+ M\left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{y}{1+y} - \frac{y_0}{1+y_0}\right|^{\alpha_2}$$

It is obvious that $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for $0 < \alpha \leq 1$ and $\forall a, b \geq 0$. So

$$\begin{aligned} &\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{s}{1+s} - \frac{y_0}{1+y_0}\right|^{\alpha_2} \\ &\leq \left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha_1}\left|\frac{s}{1+s} - \frac{y}{1+y}\right|^{\alpha_2} \\ &+ \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{y}{1+y} - \frac{y_0}{1+y_0}\right|^{\alpha_2} \end{aligned}$$

Using Hölder inequality with $p_1 = \frac{2}{\alpha_1}$, $p_1' = \frac{2}{2-\alpha_1}$ and $p_2 = \frac{2}{\alpha_2}$, $p_2' = \frac{2}{2-\alpha_2}$, the proof is done.

Remark 8 *If it is taken* $E = R_+$ *as a special case of Theorem 7, since* $d(x,E) = 0$ *and* $d(y,E) = 0$*, the following result can be obtained:*
$\forall f \in W_{\alpha_1,\alpha_2,R_+^2}$

$$|L_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)| \leq$$
$$M\left[(\delta_1(x))^{\alpha_1}(\delta_2(y))^{\alpha_2}\right].$$

where $\delta_1(x)$ *and* $\delta_2(y)$*; defined as in (8).*

5 Conclusion

The approximation properties of generalization of q-Bleimann, Butzer and Hahn operators with n variables can also be done in a similar way.

Acknowledgements: The author would like to thank to Professor Ogun Dogru for his valuable suggestions and remarks during the preparation of this work.

References:

[1] Bleimann, G. and Butzer, P. L. and Hahn, L., A Bernstein-type operator approximating continuous functions on the semi-axis, Math. Proc. A 83 (1980) 255-262

[2] Gadjiev, A. D. and Çakar, Ö., On uniform approximation by Bleimann, Butzer anh Hahn operators on all positive semi-axis, Trans. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. 19 (1999) 21-26

[3] Phillips, George M., Bernstein polynomials based on the q-integers, The heritage of P.L.Chebyshev: Ann. Numer. Math. 4 (1997) 511-518

[4] Aral, Ali and Doğru, Ogün, Bleimann Butzer and Hahn operators based on q-integers, Journal of Inequalities and Applications, (to appear)

[5] Andrews, G. E., Askey, R. and Roy,R, Special Functions, Cambridge University Press, 1999

[6] D.D. Stancu, A new class of uniform approximating polynomial operators in two and several variables, Proc. Conf. on Constructive Theory of Functions, Budapest, 1969, 443-455

[7] Volkov, V. I., On the convergence of sequences of linear positive operators in the space of continuous functions of two variables, Dokl. Akad. Nauk SSSR (N.S.), 115, 1957, 17-19