On Competition Between Modes of the Onset of Marangoni Convection with Free-slip Bottom under Magnetic Field

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Abstract: In this paper we use a numerical technique to analyze the onset of Marangoni convection in a horizontal layer of electrically-conducting fluid heated from below and cooled from above in the presence of a uniform vertical magnetic field. The top surface of a fluid is deformably free and the bottom boundary are rigid and free-slip. The critical values of the Marangoni numbers for the onset of Marangoni convection are calculated and the latter is found to be critically dependent on the Hartmann , Crispation and Bond numbers. In particular we present an example of a situation in which there is competition between modes at the onset of convection.

Key–Words: Marangoni Convection, Magnetic Field, Free-slip

1 Introduction

Convection in a plane horizontal fluid layer heated from below, known as the Rayleigh-Bénard convection, is the type of convection considered most frequently. Rayleigh [1] was the first to solve the problem of the onset of thermal convection in a horizontal layer of fluid heated from below. His linear analysis showed that Bénard convection occurs when the Rayleigh number exceeds a critical value. Theoretical analysis of Marangoni convection was started with the linear analysis by Pearson [2] who assumed an infinite fluid layer, a nondeformable case and zero gravity in the case of no-slip boundary conditions at the bottom. He showed that thermocapillary forces can cause convection when the Marangoni number exceeds a critical value in the absence of buoyancy forces. Pearson [2] obtained the critical Marangoni number, $M_c = 79.607$ and the critical wave number $a_c = 1.9929$. Linear stability analysis of Marangoni convection with free-slip boundary conditions at the bottom was first investigated by Boeck [3]. For free-slip case, Boeck [3] obtained the critical Marangoni number, $M_c = 57.598$ and the critical wave number $a_c = 1.7003$.

The effect of a magnetic field on the onset of steady buoyancy- and thermocapillary-driven (Bénard-Marangoni) convection in a fluid layer with a nondeformable free surface was first analyzed by Nield [4]. He found that the critical Marangoni number monotonically increased as the strength of vertical magnetic field increased. This indicates that the Lorentz force suppressed Marangoni convection. Later, the effect of a magnetic field on the onset of steady Marangoni convection in a horizontal layer of fluid has been discussed in a series of papers by Wilson [5, 6, 7]. The influence of a uniform vertical magnetic field on the onset of oscillatory Marangoni convection was treated by Hashim and Wilson [8] and Hashim and Arifin [9].

The above investigators pertain their analyses to Marangoni convection in the presence of magnetic field with no-slip lower boundary condition. In this study, we consider the onset of steady Marangoni convective instability in a horizontal fluid layer of electrically-conducting fluid with a deformable upper free surface and a free-slip lower surface, subject to a uniform magnetic field. To the author's best knowledge, this problem has not been reported in the literature. The linear stability theory is applied and the resulting eigenvalue problem is solved numerically. The effects of the Hartmann number and a free surface deformation on the onset of steady Marangoni convection are studied.

2 Problem formulation

Consider a horizontal fluids layer of depth d heated from below subject to a uniform vertical magnetic field and a uniform vertical temperature gradient. The fluid layer is bounded below by a horizontal solid boundary at constant temperature T_1 and above by a free surface at constant temperature T_2 which is in

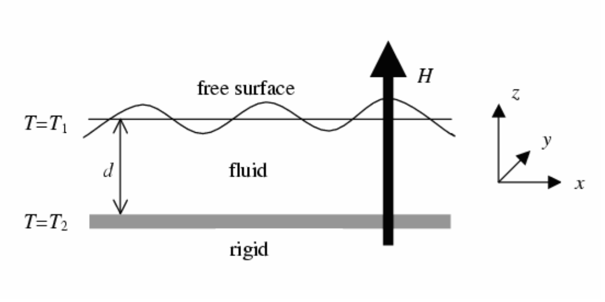

Figure 1: Geometry of the unperturbed state

contact with a passive gas at constant pressure P_0 and constant temperature T_∞. We used cartesian coordinates with two horizontal x- and y-axes are located at the lower solid boundary and a positive z-axis is directed toward the free surface. The surface tension, τ is assumed to be a linear function of the temperature

$$\tau = \tau_0 - \gamma\,(T - T_0), \tag{1}$$

where τ_0 is the value of τ at temperature T_0 and the constant γ is positive for most fluids. The density of the fluids is given by

$$\rho = \rho_0\{1 - \alpha(T - T_0)\}, \tag{2}$$

where α is the positive coefficient of the thermal liquid expansion and ρ_0 is the value at the reference temperature T_0.

Subject to the Boussinesq approximation the governing equations for an incompressible, ellectrically conducting fluid in the presence of a magnetic field are

$$\nabla\cdot\mathbf{U} = 0, \tag{3}$$
$$\nabla\cdot\mathbf{H} = 0, \tag{4}$$
$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{U} = -\frac{1}{\rho}\nabla\Pi + \nu\nabla^2\mathbf{U} + \frac{\mu}{4\pi\rho}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{5}$$
$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{H} = (\mathbf{H}\cdot\nabla)\mathbf{U} + \eta\nabla^2\mathbf{H}, \tag{6}$$
$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)T = \kappa\,\nabla^2 T. \tag{7}$$

where $\mathbf{U}$ is the fluid velocity, $\mathbf{H}$ is the magnetic field, T is the temperature and $\Pi = p + \mu|\mathbf{H}|^2/8\pi$ is the magnetic pressure, where p is the fluid pressure. When motion occurs the upper free surface of the layer will be deformable with its position at $z = d + f(x, y, t)$. At the free surface, we have the usual kinematic condition together with conditions of continuity of the normal and tangential stresses, and for the temperature obeys Newton's law of cooling, -$k\partial T/\partial\mathbf{n} = h(T - T_\infty)$, where k and h are the thermal conductivity of the fluid and the heat transfer coefficient between the free surface and the air, respectively, and $\mathbf{n}$ is the outward unit normal to the free surface. At the lower rigid boundary the usual no-slip conditions requires continuity of velocity between the solid and the fluid.

To simplify the analysis, it is convenient to write the governing equations and boundary conditions in a dimensionless form. In the dimensionless formulation, scales for length, time, velocity, temperature and magnetic field have been taken to be d, d^2/ν, ν/d, $\beta d\nu/\kappa$ and $\mu\overline{H}/\eta$ respectively. Furthermore, six dimensionless groups appearing in the problem are the Marangoni number $M = \gamma\beta d^2/\rho\nu\kappa$, the Hartmann number (the square root of the Chandrasekhar number) $H = \mu\overline{H}d(\sigma/\rho\nu)^{1/2}$, the Biot number $B_{\rm i} = hd/k$, the Bond number $B_{\rm o} = \rho g d^2/\tau_0$, the Prandtl number $P_1 = \nu/\kappa$ and the magnetic Prandtl number $P_2 = \nu/\eta$.

3 Linearized problem

The linearized equations and boundary conditions governing the onset of Marangoni convection in an initially quiescent horizontal fluid layer bounded above by a deformable free surface and bounded below by a thermally conducting planar boundary subject to a uniform vertical magnetic field and a uniform temperature gradient have been obtained by several authors (see, for example, Hashim and Arifin [9]) and are given by

$$(D^2 - a^2)T + w = 0, \tag{8}$$

$$\left[(D^2 - a^2)^2 - H^2 D^2\right] w = 0, \tag{9}$$

subject to

$$w = 0, \tag{10}$$

$$P_1 C_{\rm r}[(D^2 - 3a^2 - H^2 - s)Dw] - a^2(a^2 + B_{\rm o})f = 0, \tag{11}$$

$$P_1(D^2 + a^2)w + a^2 M(P_1 T - f) = 0, \tag{12}$$

$$h_z = 0, \tag{13}$$

$$P_1 DT + B_{\rm i}(P_1 T - f) = 0, \tag{14}$$

evaluated on the undisturbed position of the upper free surface $z = 1$, and

$$w = 0, \tag{15}$$

$$D^2 w = 0, \tag{16}$$

$$h_z = 0, \tag{17}$$

$$T = 0, \tag{18}$$

on $z = 0$. The operator $D = \mathrm{d}/\mathrm{d}z$ denotes differentiation with respect to the vertical coordinate z. The variables w, T, h_z and f denote respectively the vertical variation of the z-velocity, temperature, magnetic field and the magnitude of the free surface deflection of the linear perturbation to the basic state with total wave number a in the horizontal x-y plane and complex growth rate s.

4 Solution of the Linearized problem

In the general case $s = 0$, we follow the solution approach of Hashim and Wilson [8] and seek asymptotic solutions for w, T in the forms

$$w(z) = ACe^{\xi z}, \quad T(z) = Ce^{\xi z}, \tag{19}$$

where the exponent ξ and the complex contants A and C are to be determined. Substituting these forms into the Eqs. (8) and (9) and eliminating A and C we obtain a sixth-order algebraic equation for ξ, namely

$$(\xi^2 - a^2)\left[(\xi^2 - a^2 - s)^2 - H^2\xi^2\right] = 0, \tag{20}$$

with six distinct roots, which we denote by $\xi_1, \ldots, \xi_6$. where the values of $\xi_1, \ldots, \xi_6$ are solutions of the fourth-order algebraic equation

$$(\xi^2 - a^2 - s)^2 - H^2\xi^2 = 0, \tag{21}$$

while $\xi_5 = a$ and $\xi_6 = -a$. Denoting the values of A and C corresponding to ξ for $i = 1, \ldots, 6$ by A_i and C_i, respectively, we can use Eq (9) to determine A_i. We can use Eq.(11) to eliminate the free surface deflection

$$f = \frac{P_1 C_\mathrm{r}\,(D^2 - 3a^2 - H^2)\,Dw}{a^2(a^2 + B_\mathrm{o})}, \tag{22}$$

evaluated on $z = 1$, leaving the six boundary conditions (11), to determine the six unknowns $C_1, \ldots, C_6$, and the general solution to the stability problem therefore

$$w(z) = \sum_{i=1}^{6} A_j C_j e^{\xi_j z}, \quad T(z) = \sum_{j=1}^{6} C_j e^{\xi_j z}. \tag{23}$$

The dispersion relation between M, a, C_r, H^2, B_o and B_i is determined by substituting these solutions into boundary conditions and evaluating the resulting 6×6 real determinants of the coefficients of the unknowns, which can be written in the form $M = -D_1/D_2$, where the two 6×6 real determinants D_1 and D_2 are independent of M.

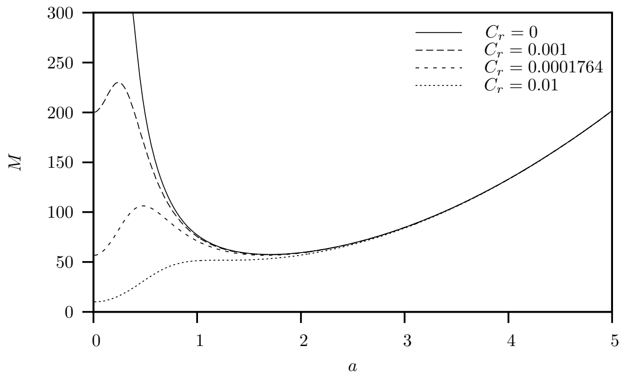

Figure 2: Numerically-calculted Marangoni number, M as a function of the wave number, a, for various values of Crispation numbers, C_r in the case $H = 0$, $B_\mathrm{i} = 0$ and $B_\mathrm{o} = 0.1$.

5 Results

The effect of a magnetic field on the onset of Marangoni convection in a fluid layer with free-slip bottom in the case of a deformable free surface ($C_\mathrm{r} \neq 0$) is investigated numerically. Before presenting the numerical results, it is helpful to specify the range for parameters B_i, B_o and C_r which are respectively given by $10^{-3} \le B_\mathrm{i} \le 10^{-1}$, $10^{-3} \le B_\mathrm{o} \le$ and $10^{-6} \le C_\mathrm{r} \le 10^{-2}$ for most fluids layers of depths ranging from 0.01 cm to 0.1 cm and are in contact with air (see Palmer and Berg (1972)). All numerical calculations reported in this paper are done for the case $B_\mathrm{i} = 0$, $B_\mathrm{o} = 0.1$ and $P_1 = 1$.

Figure 2 shows the numerically-calculated steady marginal stability curves plotted for different values crispation number C_r in the case $H = 0$. The crispation number C_r, associated with the inverse effect of the surface tension, represents the degree of the free surface deformability. When C_r becomes large (corresponding to weak surface tension), the marginal curve has global minimum at zero wavenumber. In contrast, for small values of C_r, the marginal curve has global minimum at nonzero wavenumber. At some transition value of C_r, the marginal curve has two local minima that is one at zero wave number and the other at nonzero wave number. The transition value of C_r for the case shown in Fig. 2 is $C_\mathrm{r} \simeq 0.0001764$. For C_r greater than 0.0001764, the wave number at marginal stability suddenly jump from nonzero number to zero. Similar competition between different modes was identified by Hashim and Arifin [9] in the case no-slip condition.

Figure 3 shows the numerically-calculated Marangoni number, M as a function of the wavenumber, a for different values of the Hartmann number, H in the case $C_\mathrm{r} = 0$. From Fig. 3 it is seen that

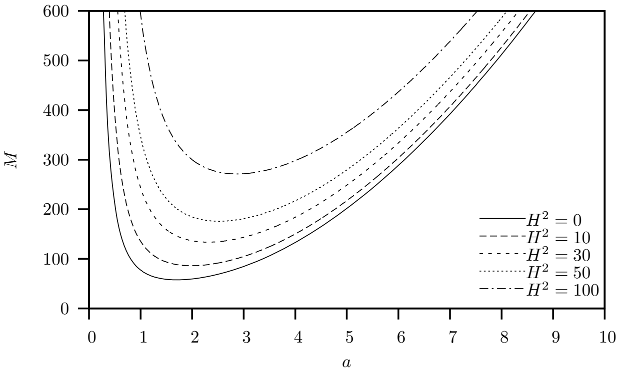

Figure 3: Numerically-calculted Marangoni number, M as a function of the wavenumber, a, for a various values of Hartmann numbers, H in the case $C_\mathrm{r} = 0$, $B_\mathrm{i} = 0$ and $B_\mathrm{o} = 0.1$.

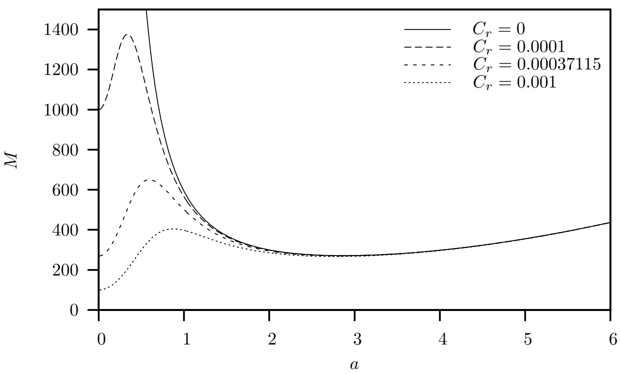

Figure 4: Numerically-calculted Marangoni number, M as a function of the wavenumber a, for various values of Crispation numbers, C_r in the case $H^2 = 100$, $B_\mathrm{i} = 0$ and $B_\mathrm{o} = 0.1$.

the critical Marangoni number increase with an increase of the Hartmann number. Thus, the effect of magnetic field make the system become more stable. Numerically-calculated Marangoni number, M as a function of the wavenumber, a for different values of the $C_\mathrm{r} \neq 0$ in the case $H^2 = 300$ are shown in Fig. 4. The figure shows parts of the marginal stability curves in the case $C_\mathrm{r} = 0.00037115$ and $H^2 = 300$ in which zero mode (infinite wavelength) and nonzero mode (finite wavelength) occur simultaneously at the onset of convection.

Figure 5 shows the numerically-calculated Marangoni number, M as a function of the wavenumber, a for different values of the Hartmann number, H in the case $C_\mathrm{r} = 0.001$. In this case, the marginal stability curve have a global minimum at the nonzero value of a without a magnetic field. But, the marginal stability curve always have a global minimum at zero value in the limit of large magnetic field. We also found two steady modes occur simultaneously at the onset of convection when $H^2 = 300$.

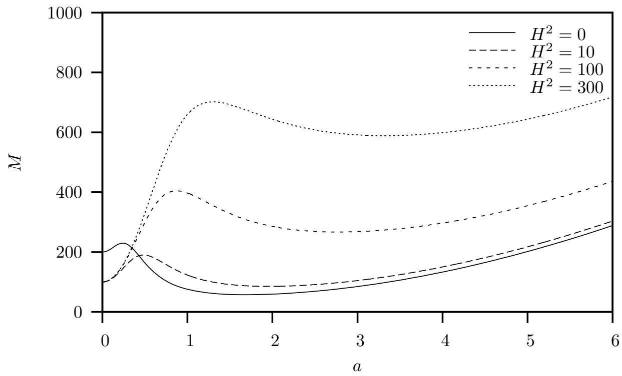

Figure 5: Numerically-calculted Marangoni number, M as a function of the wavenumber, a, for various values of Hartmann numbers, H in the case $C_\mathrm{r} = 0.001$, $B_\mathrm{i} = 0$ and $B_\mathrm{o} = 0.1$.

6 Conclusions

The effect of the magnetic field on the onset of steady Marangoni convection in a horizontal layer of electrically-conducting fluid which is free above and rigid below with free-slip condition has been studied. If the free surface is nondeformable, the absence of a magnetic field always has the stabilizing effect of increasing the critical Marangoni number for the onset of steady convection. If the free surface is deformable, then all the marginal stability curves have two local minima. The linear analysis presented in this work revealed a situation in which two steady modes compete at the onset of convection.

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