Fuzzy chromatic number and fuzzy de£ning number of certain fuzzy graphs

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Abstract: In this paper we introduce the concepts of fuzzy de£ning number in some fuzzy graphs. we determine the chromatic number for several classes of fuzzy graphs and obtain bounds for them. We also determine the fuzzy de£ning number of the classes.

Key–Words: fuzzy graph, fuzzy chromatic number, union, join, the Cartesian product.

1 Introduction

The motivation for introducing fuzzy graphs is re¤ected upon by Rosenfeld in (Zadeh, 1975). Then the fuzzy graphs were discussed in (Kaufman, 1977, Dubois, Prade, 1980, Volkmann, 1991) and in some others papers and books. There are many problems, which can be solved with the help of the fuzzy graphs. Graph theory has proved to be an extremely useful tool for solving many tasks (Berge, 1989), [**?**]. Let V be a £nite nonempty set. Let E be the collection of all two-element subsets of V $(E \subseteq V \times V)$. A fuzzy graph $\tilde{G} = (\sigma, \mu)$ is a set with two functions $\sigma : V \to I$ and $\mu : E \to I'$ such that $\mu(\{x, y\}) \preceq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. Where $\sigma : V \to I$ and $\mu : E \to I'$ are the vertex membership function on the I and edge membership function on the I' respectively, and $\sigma(x), \mu(\{x, y\})$ re¤ects the ambiguity of the assertion x belongs to I, I' respectively, [2]. In classical fuzzy-set theory the set I is usually de£ned as the closed interval $[0, 1]$, in such a way that $\sigma(x) = 0$ indicates that x does not belong to I, $\sigma(x) = 1$ indicates that x strictly belongs to I, and any intermediate value represents the degree in which x could belong to I. However, the set I could be a discrete set of the form $I = \{0, 1, ..., k\}$, where $\sigma(x) \leq \sigma(x')$ indicates that the degree of membership function x to I is lower than the degree of membership function x' . In general, the set I can be any ordered set, not necessarily numerical, for instance, $I = \{n, l, m, h, t, c, p\}$, where n, l, m, h, t, c and p denote the fuzzy degrees null, low, medium, high, total, complete and perfect, respectively, [8].

Graph theory has numerous applications to problems in systems analysis, operations research,mobile telephone service,missile guidance , transportation, and economics. In many cases, however, some aspects of the graph-theoretic problem are uncertain. In these cases, it can be useful to deal with this uncertainty using the methods of fuzzy logic. Graph coloring is one of the most studied problems of combinatorial optimization. An important area of application of the coloring problem is management science,[3, 5, 9, 12, 13]..

1.1 Preliminaries

De£nition 1 *Let S be the available color set. Let d be the* ***dissimilarity measure*** *de£ned by $d : S \times S \to [0, \infty)$ with the following properties:*

1. $d(r, s) \geq 0 \quad \forall r, s \in S,$
2. $d(r, s) = 0 \Leftrightarrow r = s \quad \forall r, s \in S,$
3. $d(r, s) = d(s, r) \quad \forall r, s \in S.$

This dissimilarity measure d can take into account the incompatibility degree in the sense that the more incompatible two vertices are, the more distant their associated colors. In this way, an extended coloring function is introduced [8].

De£nition 2 *Let I', be a domain of the edge membership function of the fuzzy graph $\tilde{G} = (\sigma, \mu)$, we de£ned $f : I' \to [0, \infty)$ be a non-negative, non-decreasing (with respect to the order $\leq$) and real scale function, i.e.*

$f(\mu_1) \leq f(\mu_2)\ \forall \mu_1, \mu_2 \in I'$ such that $\mu_1 \leq \mu_2$ (2).

De£nition 3 *Given I, be a domain of the vertex membership function of the fuzzy graph $\tilde{G} = (\sigma, \mu)$, we de£ned $g : I \times I \to [0, \infty)$ be a non-negative, non-decreasing (with respect to the order $\leq$) real scale function, i.e.*

$\forall \sigma_i, \sigma_j, \sigma_{i'}, \sigma_{j'} \in I$ such that $\mu_{ij} \leq \mu_{i'j'}$ $g(\sigma_i, \sigma_j) \leq g(\sigma_{i'}, \sigma_{j'})$ (3)

De£nition 4 *Given a fuzzy graph $\tilde{G} = (\sigma, \mu)$, a color set S, a dissimilarity measure d de£ned on S and scale functions f, g, a (d, f, g)-extended coloring function of G , denoted as $C_{d,f,g}$, for short as C, is a mapping: $C : V \to S$ with the property:*

$d(C(i), C(j)) \geq f(\mu_{ij}) \forall i, j \in V$, such that $i \neq j$,

$d(C(i), C(j)) \geq g(\sigma_i, \sigma_j) \forall i, j \in V$, such that $i \neq j$.

(4)

A (d, f, g)-extended k-coloring $C^k_{d,f,g}$, for short C^k, is a (d, f, g)-extended coloring function with no more than k different colors: $C^k : V \to S$, where $S = \{1, ..., k\}$.

De£nition 5 *Given a fuzzy graph $\tilde{G}$, a dissimilarity measure d and scale functions f, g, the* ***minimum value*** *k for which a (d, f, g)-extended k-coloring of $\tilde{G}$ exists is called the (d, f, g)-chromatic number of $\tilde{G}$ and it is denoted by $\chi_{d,f,g}(\tilde{G})$ [8].*

De£nition 6 *Given a fuzzy graph $\tilde{G} = (\sigma, \mu)$, a set of vertices S with an assignment of colors to them is called a de£ning set of the vertex coloring of $\tilde{G}$ if there exists a unique extension of the colors of S to a $\chi_{d,f,g}(\tilde{G})$- coloring of the vertices of $\tilde{G}$. A de£ning set with minimum cardinality, is called a minimum de£ning set and its cardinality, the de£ning number, is denoted by $d_{d,f,g}(\tilde{G}, \chi_{d,f,g})$, for short $d_{d,f,g}$ [7].*

2 Some results

De£nition 7 *Let $\sigma : V \longrightarrow I$ be a fuzzy subset of V. Then the* ***complete fuzzy graph*** *on σ is de£ned to be (σ, μ) where $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $xy \in E$ and is denoted by K_σ [11]..*

Proposition 8 *Let K_σ be a complete fuzzy graph, and d be a dissimilarity measure and f, g be the scale functions where $g(\sigma(i), \sigma(j)) = f(\mu_{ij})$ for all $i, j \in V$. Then $1 \leq \chi_{d,f,g}(K_\sigma) \leq 1 + (n - 1)D$ where $D = Max\{f(\mu(xy)) \mid xy \in E\}$ and $n = |V|$.*

Proof: By de£nition any edge of K_σ is an effective edge. Consider the greedy coloring, if for v_iv_j, $D = f(\mu(v_iv_j))$ then by rearrangement in number of vertices we put $D = f(\mu(v_1v_2))$. Now we can color the £rst vertex by number 1, the second vertex we can color by number $1 + D$ then third vertex is colored by $1 + 2D$ since $D \geq f(\mu(3y))\ \forall y \in V$. Also the forth vertex is colored with the number $1 + 3D$ because $D \geq f(\mu(4y))$, In this way we found the n_{th} vertex color by $1+(n-1)D$. This coloring is an upper bound for coloring of this graph, because we suppose D is a maximum value scale function for every edge. □

De£nition 9 *A* ***path*** *P a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct vertices $(u_0, u_1, u_2, ..., u_n)$ such that $\mu(u_i, u_{i+1}) > 0$, $1 \leq i \leq n$ and n is called the* ***length*** *of P. The path P is called $u_0 - u_n$ path [11].*

Proposition 10 *Let $\tilde{P}$ be a fuzzy path, a d be dissimilarity measure and f, g be the scale functions where $g(\sigma(i), \sigma(j)) = f(\mu_{ij})$ for all $i, j \in V$. Then $1 \leq \chi_{d,f,g}(\tilde{P}) \leq 1+D$ if $D = max\{f(\mu(xy)) \mid xy \in E\}$. Also $d_{d,f,g}(P) = n - 1$.*

Proof: If $D = f(\mu(v_iv_{i+1}))$ then the vertex v_i is colored by color 1 and the vertex $v_{i+1}$is colored by $1 + D$. Also for coloring other vertices we assign vertices v_j,v_{j+1} for all $j > i + 1$ by color 1 and $1 + D$ respectively, and for all $j < i + 1$ by color $1 + D$ and 1 respectively, therefore the chromatic number of P is $1 + D$. By greedy coloring. Since for coloring fuzzy path we determined an edge with maximum value scale function f, denoted as D and coloring all vertices by color 1 and $1+D$. Also we can color fuzzy Path by some colors such that the relation in condition (4) on de£nition 4 is true. Then we must choose $n - 1$ vertices for forcing color of the last vertex. So $d_{d,f,g}(P) = n - 1$. . □

De£nition 11 *A fuzzy graph $G = (\sigma, \mu)$ is said to be* ***bipartite*** *if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $x \in V_1$ and $y \in V_2$ then G is called a* ***fuzzy complete bipartite graph*** *and is denoted by K_{σ_1,σ_2} where σ_1 and σ_2 are, respectively, the restrictions of V to V_1 and V_2 [2].*

Proposition 12 *Let K_{σ_1,σ_2} be a fuzzy complete bipartite graph, d a dissimilarity measure and the f, g be scale functions. where $g(\sigma_1(i), \sigma_2(j)) = f(\mu_{ij})$*

for all $i, j \in V$. Then $1 \leq \chi_{d,f,g}(K_{\sigma_1,\sigma_2}) \leq 1 + D$ if $D = max\{f(\mu(xy)) \mid xy \in E\}$. Also $d_{d,f,g}(K_{\sigma_1,\sigma_2}) = n - 1$.

Proof: Consider the greedy coloring. If $D = f(\mu(uv))$ where $u \in V_1$ and $v \in V_2$ then the vertex u is colored by color 1 and vertex v is colored by $1+D$. For coloring other vertices we assign the color 1 to the $v_i \in V_1$ and the color $1 + D$ to the vertices $v_j \in V_2$. (Also we can color this by some colors such that the relation in (4) is true) therefore the chromatic number of K_{σ_1,σ_2} is $1 + D$. By the de£nition of K_{σ_1,σ_2}, $D = max\{f(\mu(xy)) \mid xy \in E\}$, by the greedy coloring, Since we can color vertices of part one in K_{σ_1,σ_2} by color 1 and vertices part two by color $1 + D$. Also we can color this vertices by some colors such that the relation in condition (4) on de£nition 4 is true. Therefore we must choose $n-1$ vertices for forcing coloring the last vertex. So $d_{d,f,g}(K_{\sigma_1,\sigma_2}) = n - 1$. . □

Proposition 13 *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$, d a dissimilarity measure and the f, g be scale functions, where $g(\sigma_t(i), \sigma_t(j)) = f(\mu_{ij})$ for all $i, j \in V, t = 1, 2$. If $G = (G_1 \cup G_2)$ then $1 \leq \chi_{d,f,g}(G) \leq max\{\chi_{d,f,g}(G_1), \chi_{d,f,g}(G_2)\}$. Also $d_{d,f,g}(G_1 \cup G_2) = d_{d,f,g}(G_1) + d_{d,f,g}(G_2)$.*

Proof: Let $D = max\{\chi_{d,f,g}(G_1), \chi_{d,f,g}(G_2)\}$. We consider the greedy coloring, If the fuzzy graph $G_1 = (\sigma_1, \mu_1)$ is colored by $\chi_{d,f,g}(G_1)$ colors and $G_2 = (\sigma_2, \mu_2)$ is colored with$\chi_{d,f,g}(G_2)$. By the de£nition of $G = (G_1 \cup G_2)$, we color vertices $v_i \in V_1$ by D colors and for coloring vertices $v_j \in V_2$ with D colors, therefore the chromatic number of $G_1 \cup G_2$ is D. If the de£ning number of fuzzy graph $G_1 = (\sigma_1, \mu_1)$ is $d_{d,f,g}(G_1)$ and the de£ning number of $G_2 = (\sigma_2, \mu_2)$ is $d_{d,f,g}(G_2)$ then we can choose $d_{d,f,g}(G_1)$vertices of G_1 and $d_{d,f,g}(G_2)$vertices of G_2 therefore the de£ning number of $G_1 \cup G_2$ is obtained $d_{d,f,g}(G_1) + d_{d,f,g}(G_2)$. . □

Proposition 14 *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$, d a dissimilarity measure and the f, g be scale functions, where $g(\sigma_t(i), \sigma_t(j)) = f(\mu_{ij})$ for all $i, j \in V, t = 1, 2$. If $G = (G_1 + G_2)$ then $1 \leq \chi_{d,f,g}(G) \leq \chi_{d,f,g}(G_1) + \chi_{d,f,g}(G_2) + D$, where $D = f(max\{\sigma_i \wedge \sigma_j, \forall i \in V_1, j \in V_2\})$.*

Proof: We consider the greedy coloring, If the chromatic number of fuzzy graph $G_1 = (\sigma_1, \mu_1)$ is $\chi_{d,f,g}(G_1)$ and the chromatic number fuzzy graph G_2 is $\chi_{d,f,g}(G_2)$ then by the de£nition of $G = (G_1+G_2)$, if the vertex v_n is the last vertex in G_1 is colored, then we color vertex $v_1 \in V_2$ with $D + \chi_{d,f,g}(G_1)$ and the other vertices in G_2 are color by the greedy coloring. Then the fuzzy graph G is colored by $\chi_{d,f,g}(G_1) + \chi_{d,f,g}(G_2) + D$ (since $s = f(\sigma_i \wedge \sigma_j) \leq D$ also we can color vertex $v_1 \in V_2$ by color $s + \chi_{d,f,g}(G_1)$ where the relation (4) is true.) there for the chromatic number of G is at most $\chi_{d,f}(G_1) + \chi_{d,f}(G_2) + D$. . □

In what follows the de£nition of fuzzy graph $G = (G_1 \Box G_2)$, we will call the $n = |V_1|$ copies of G_2 in the Cartesian product the rows of the $G = (G_1 \Box G_2)$, and the $m = |V_2|$ copies of G_1 the columns of the $G = (G_1 \Box G_2)$.

Proposition 15 *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively with $V_1 \cap V_2 = \phi$, d a dissimilarity measure and the f, g be scale functions, where $g(\sigma_t(i), \sigma_t(j)) = f(\mu_{ij})$ for all $i, j \in V, t = 1, 2$. If $G = (G_1 \Box G_2)$ then $1 \leq \chi_{d,f,g}(G) \leq \chi_{d,f,g}(G_1) + \chi_{d,f,g}(G_2)$.*

Proof: We consider the greedy. If the chromatic number of fuzzy graph $G_1 = (\sigma_1, \mu_1)$ is $\chi_{d,f,g}(G_1)$ and the chromatic number fuzzy graph G_2 is $\chi_{d,f,g}(G_2)$. We assign the colors $\{1, 2, ..., \chi_{d,f,g}(G_2)\}$ the vertices v_{1j} for $j = 1, ..., m$. (since for any vertex we have a copy of G_1) and if the vertex v_{1m} is last vertex where colored by color $\chi_{d,f,g}(G_2)$ then we color a copy of G_1 corresponding with v_{1m} and the vertices v_{1m}, v_{2m}, v_{3m}, ..., v_{nm} are colored by $\chi_{d,f,g}(G_2) + 1$, $\chi_{d,f,g}(G_2) + 2$, $\chi_{d,f,g}(G_2) + 3$, ..., $\chi_{d,f,g}(G_2) + \chi_{d,f,g}(G_1)$, respectively. Then the chromatic number fuzzy graph G is $\chi_{d,f,g}(G_1) + \chi_{d,f,g}(G_2)$. . □

De£nition 16 *A **cycle** C in a fuzzy graph $\tilde{G} = (\sigma, \mu)$ is a sequence of distinct vertices $(u_0, u_1, u_2, ..., u_n)$ such that $\mu(u_i, u_{i+1}) > 0, 1 \leq i \leq n$ where $u_0 = u_n$ and n is called the **length** of C, and is denoted by $\tilde{C}_n$.*

Lemma 17 *Let $\tilde{C}_n$ be a fuzzy cycle , d a dissimilarity measure and the f, g be scale functions, where $g(\sigma_t(i), \sigma_t(j)) = f(\mu_{ij})$ for all $i, j \in V, t = 1, 2$. Then $1 \leq \chi_{d,f,g}(\tilde{C}_n) \leq 1 + D$ if n is even and $1 \leq \chi_{d,f,g}(\tilde{C}_n) \leq 1 + 2D$ if n is odd where $D = max\{f(\mu(xy)) \mid xy \in E\}$.*

Proof: First assume that n is even, If $D = f(\mu(u_i u_{i+1})$, without loss generality we can assume that $D = f(\mu(u_1u_2)$ then the £rst vertex colored by color 1 and the second vertex is colored by $1 + D$ and the other vertices color by $1, 1 + D$ respectively. Since n is even the n_{th} vertex colored by $1 + D$ therefore its chromatic number in this case is $1 + D$. If assume that n is odd, if $D = f(\mu(u_iu_{i+1})$, without loss generality we can assume that $D = f(\mu(u_1u_2)$ then the £rst vertex is colored by color 1 and the second vertex is colored by $1 + D$ and the other vertices are colored by 1 and $1 + D$ respectively. Since n is odd, the $(n - 1)_{th}$ vertex colored by $1 + D$ and can't color the n_{th} vertex by color 1 because its neighborhood has color 1 therefore we color it by color $1 + 2D$. So the chromatic number in this case is $1 + 2D$. $\square$

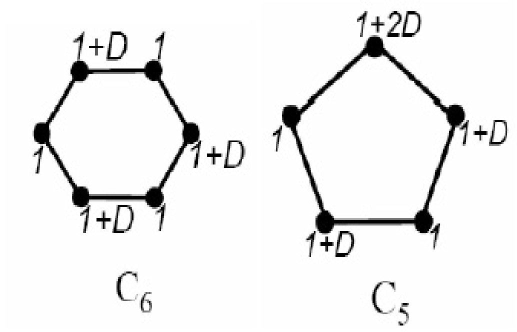

Fig.1. Fuzzy cycles $\tilde{C}_6$ and $\tilde{C}_5$

De£nition 18 *A $n \square m$ fuzzy torus is de£ned as $\tilde{C}_n \square \tilde{C}_m$, where $\tilde{C}_n$ and $\tilde{C}_m$ are fuzzy cycles of length n and m, respectively, and the multiplication is the fuzzy Cartesian product for graphs.*

In what follows, the m copies of $\tilde{C}_n$ in the fuzzy Cartesian product [10] are the rows of the torus, and the n copies of $\tilde{C}_m$ are the columns of the torus.

Lemma 19 *Let $\tilde{C}_n \square \tilde{C}_m$ be a fuzzy torus, d a dissimilarity measure and the f, g be scale functions, where $g(\sigma_t(i), \sigma_t(j)) = f(\mu_{ij})$ for all $i, j \in V, t = 1, 2$. Then $1 \leq \chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1 + D$ if n, m are even and $1 \leq \chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1 + 2D$ if n or m is odd where $D = max\{f(\mu(xy)) \mid xy \in E\}$.*

Proof: If m, n both are even, by de£nition of fuzzy Cartesian product we have the m copies of $\tilde{C}_n$ in the rows and the n copies of $\tilde{C}_m$ in the columns if by proof of proposition 8 we color the £rst copy of $\tilde{C}_n$ in row where the vertex v_{11} is colored by color 1 and color the £rst column in fuzzy torus where contains v_{11} by colors $1, 1+D$ respectively. Therefore in other rows one vertex is colored and other vertex in any rows is colored too. So we have $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1 + D$. If one of the m, n are odd then by de£nition of fuzzy Cartesian product and without loss generality assume m is odd and n is even then the £rst vertex of £rst rows as denoted v_{11} colored by color 1 and other vertices this row by way of proof proposition 8 is colored by $1 + D, 1$ respectively. Since m is odd we can coloring the £rst column by $1, 1 + D, 1 + 2D$ respectively. Also the £rst vertex of other rows is colored so we can colored them by way in proof of proposition 8 therefore in this case $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1 + 2D$. If n, m both are odd by consider greedy coloring and fuzzy Cartesian product we colored the £rst copies of $\tilde{C}_n$ in rows by color $1, 1 + D, 1 + 2D$ by way in proposition 8 then the £rst vertex of £rst copies of $\tilde{C}_m$ in column is colored by 1 and other vertices this column is coloring by $\{1 + D, 1 + 2D\}$ therefore for coloring any vertices of fuzzy torus in this state since two vertices in above and left it is colored in the worst state if two vertices have different colors we can colored it vertex by another color. So we can colored fuzzy torus by $\{1 + D, 1 + 2D\}$ or $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1+2D$. in the worst state if all vertices membership value and edge membership value are equal then $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) = 1 + 2D$, .. $\square$

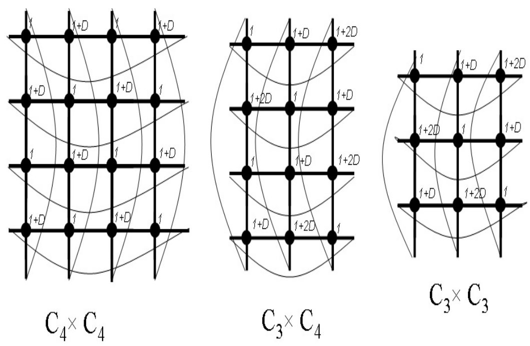

Fig.2. Fuzzy torus $\tilde{C}_4 \square \tilde{C}_4$ and $\tilde{C}_4 \square \tilde{C}_3$ and $\tilde{C}_3 \square \tilde{C}_3$.

De£nition 20 *A fuzzy hypercube $\tilde{Q}_n$ of dimension n is de£ned as $\tilde{Q}_{n-1} \square \tilde{K}_2$, where $\tilde{K}_2$ is a fuzzy complete graph with 2 vertices, and $\tilde{Q}_{n-1}$ is fuzzy hypercube of dimension $n - 1$ with 2^{n-1} vertices and $\tilde{Q}_1 = \tilde{K}_2$.*

Proposition 21 *Let $\tilde{Q}_n$ be a fuzzy hypercube of dimension n, d a dissimilarity measure and the f, g be scale functions, Then $1 \leq \chi_{d,f,g}(\tilde{Q}_n) \leq 1 + D$ if $D = f(\mu(e)), e \in E[\tilde{K}_2]$.*

Proof: By greedy coloring, if $D = f(\mu(e))$, $e \in E[\tilde{K}_2]$, Since $\tilde{Q}_1 = \tilde{K}_2$, for $n = 1$ we can color

$\tilde{Q}_n$ by color $1, 1 + D$. For any $n \geq 2$ by de£nition Cartesian product we have 2^{n-1} copies of $\tilde{K}_2$ which membership value of vertices of $\tilde{Q}_n$ less than membership value of vertices of $\tilde{K}_2$ and edge membership value of $\tilde{Q}_n$ less than D. Therefore we can color any vertices of $\tilde{K}_2$ by $1, 1+D$ therefore we color fuzzy hypercube of dimension n where its chromatic number at last $1 + D$. In the worst state if all vertices membership value and edge membership value are equal then $\chi_{d,f,g}(\tilde{Q}_n) = 1 + D$, (see Fig.3)..

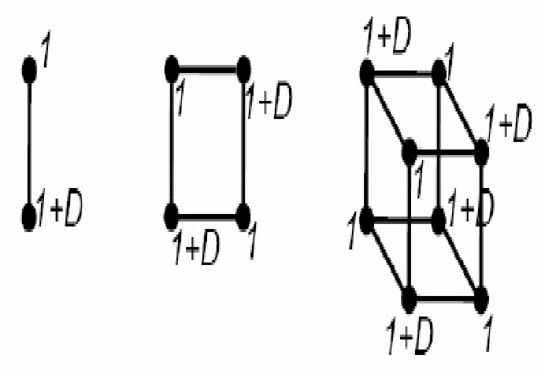

Fig.3. Fuzzy hypercube $\tilde{Q}_1$ and $\tilde{Q}_2$ and $\tilde{Q}_3$.

De£nition 22 *Let $\tilde{C}_n$ is fuzzy cycle of length n and $\tilde{P}_m$ is fuzzy path of length m, the $\tilde{C}_n \Box \tilde{P}_m$ is the fuzzy Cartesian product of them.*

Proposition 23 *Let $\tilde{C}_n \Box \tilde{P}_m$ be a fuzzy graph, d a dissimilarity measure and the f, g be scale functions, $D_1 = max\{f(\mu(xy)) \mid xy \in E[\tilde{C}_n]\}$, $D_2 = max\{f(\mu(xy)) \mid xy \in E[\tilde{P}_m]\}$. Then $1 \leq \chi_{d,f,g}(\tilde{C}_n \Box \tilde{P}_m) \leq 1 + D$ if n is even and $1 \leq \chi_{d,f,g}(\tilde{C}_n \Box \tilde{P}_m) \leq 1 + 2D$ if n is odd where $D = max\{D_1, D_2\}$.*

In what follows, the m copies of $\tilde{C}_n$ in the Cartesian product are the rows of the $\tilde{C}_n \Box \tilde{P}_m$, and the n copies of $\tilde{P}_m$ are the columns of the $\tilde{C}_n \Box \tilde{P}_m$.

Proof: If n is even by Lemma 1, we can color the fuzzy cycle $\tilde{C}_n$ by color $1, 1 + D$ and by de£nition of fuzzy Cartesian product the n copies of fuzzy path $\tilde{P}_m$ are in the column where the £rst vertex of all is colored with the color the £rst cycle in rows. Therefore for any $\tilde{P}_m$ we can color other vertices by $1, 1 + D$ or $1 + D, 1$. So in this case the chromatic number is at $1 + D$. now assume n is odd. By proposition 8 we can color the fuzzy cycle $\tilde{C}_n$ by colors $1, 1 + D, 1 + 2D$. Then we obtained the n copies of fuzzy path $\tilde{P}_m$ where one of its vertices is colored. We can color it by two colors of $\{1, 1 + D, 1 + 2D\}$ respectively. Therefore in this case chromatic number at last $1 + 2D$. also in the worst case if all vertices membership value and edge membership value are equal then $\chi_{d,f,g}(\tilde{C}_n \Box \tilde{P}_m) = 1 + 2D$. . $\Box$

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