The Zero Interval Limit Spectrum of a Truncated Fluctuation Matrix On a Univariate Function Type Multiplicative Algebraic Operator Over the Relevant Hilbert Space

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Abstract: This paper focuses on the spectral behavior of the fluctuation matrices as their construction interval varies. The fluctuation matrix definition's base operator in this work is taken an algebraic one which multiplies its operand with a univariate function remaining continous over an interval where the elements of the base Hilbert space are square integrable univariate functions. We work in the vicinity of the interval's zero limit case and obtain universal results for certain spectral properties of these types of fluctuation matrices.

Key–Words: Hermitian Operators, Expectation Value, Expectation Matrices, Fluctuation, Fluctuation Matrices

1 Introduction

Consider a Hermitian operator $\mathcal{L}$ mapping from a Hilbert Space $\mathcal{H}$ of square integrable univariate functions to itself. The expectation value of a this type operator is defined as[6]

$$\langle \phi \left| \mathcal{L} \right| \phi \rangle \equiv (\phi, \mathcal{L}\phi) \equiv \int_a^b dx \phi^*(x) \mathcal{L} \phi(x) \qquad (1)$$

where $\phi(x)$ whose norm is assumed to be 1 belongs to $\mathcal{H}$ and the second term above represents the inner product of $\mathcal{H}$. The star symbol stands for the complex conjugation. The very first term of the above equality is written in Dirac's bra and ket notation where bra and ket can be respectively considered infinite row and column vectors whose elements are indiced by a continuous variable which is not denoted explicitly in the notation. As $\phi(x)$ varies in $\mathcal{H}$ the expectation value of $\mathcal{L}$ varies between its least and greatest eigenvalues inclusive. However, there are certain subspaces of $\mathcal{H}$, where, the expectation values remain invariant. These are spanned by the eigenfunction(s) corresponding to the same eigenvalue. If there is no multiplicity in the eigenvalues these subspaces (eigenspaces) are univariate otherwise their dimensionality equals to the corresponding multiplicity because of the Hermitian nature of $\mathcal{L}$.

If we remove the condition that $\phi(x)$'s norm is equal to 1 in (1) then the above expectation value can be rewritten as the ratio of the right hand side content of (1) to the norm square of $\phi(x)$. The resulting entity is a Rayleigh quotient whose stationary directions are represented by the eigenfunctions (or eigenvectors in the Linear Vector Space Terminology) of $\mathcal{L}$ and the complex modulus of the corresponding eigenfunction(s) can be considered as probability functions which weigh the operator under consideration differently in different independent values. Therefore expectation value can be considered in fact as a probabilistic mean value, in other words, it is a statistical entity. When statistics comes to the scene we may also use statistical concepts. One of them, standart deviation in the mean values, illuminates how the situation under consideration is statistical in the nature. Now we can define the following entity to measure this nature when $\mathcal{L}$ is hermitian (if this is not the case then the first factors of $\mathcal{L}^2$ and the operator appearing in the second line below should be replaced by $\mathcal{L}$'s hermitian conjugate).

$$\begin{aligned} \mathcal{F}(\mathcal{L}, \phi) &\equiv \left\langle \phi \left| \mathcal{L}^2 \right| \phi \right\rangle - \langle \phi \left| \mathcal{L} \right| \phi \rangle^2 \\ &\equiv \langle \phi \left| \mathcal{L} \left[\mathcal{I} - \mathcal{P}_\phi \right] \mathcal{L} \right| \phi \rangle \end{aligned} \qquad (2)$$

where $\phi(s)$'s norm is assumed to be 1, $\mathcal{I}$ is the unit operator on $\mathcal{H}$ and $\mathcal{P}_\phi$ stands for the operator which projects any given function in $\mathcal{H}$ to a subspace spanned by $\phi(x)$ in the same Hilbert space. Since a projection operator and its complement to its Hilbert space is nonnegative definite, it is not hard to see that the fluctuation definition above will remain always nonnegative as ϕ browses in $\mathcal{H}$ and will vanish only on the eigenspaces of $\mathcal{L}$. In other words, this value increases up to a heighest value which is the greatest eigenvalue of the operator $\mathcal{L}\left[\mathcal{I}-\mathcal{P}_{\phi}\right]\mathcal{L}$ and diminishes down to zero with the possibility of taking the bounding values. That is, it fluctuates. This is the reason why the entity given in (2) is called fluctuation[2, 3]. More precisely speaking, its the fluctuation of the operator $\mathcal{L}$ with respect to an element $\phi(x)$ of $\mathcal{H}$.

Perhaps most easily handlable one of all operators is the operator which multiplies its operand by a univariate function since this action gives the chance of considering the operator just a plain function in the integrations. We are going to focus on these type of cases here. Although they seem to be rather simple they are very important when we need to know how smooth a function, which takes the role of an operator, is.

The rest of the paper is organised as follows. Next section covers the definition of the fluctuation matrices and their truncations. The third section deals with the spectrum of fluctuation matrices while the fourth section involves numerical implementations to illustrate the universal structure of the spectra of fluctuation matrices. The fifth section presents the concluding remarks.

2 Fluctuation Matrices and Their Truncations

In the previous section we recall the definition of the fluctuation of an operator with respect to a function chosen from the base Hilbert space and we have emphasized on the statistical nature of this entity. Since that definition is based on a continuous procedure, integration, it is better to seek the possibility of converting everything to discrete objects, that is, vectors and matrices. What we can do to proceed towards this goal is to use the expansions with respect to a basis set. Let us consider the following basis set whose elements are mutually orthogonal and have unit norms with respect to the inner product and the norm induced from this inner product in $\mathcal{H}$[4]

$$\mathcal{U} \equiv \langle u_j(x)\rangle_{j=1}^{\infty} \tag{3}$$

where x represents the independent variable and u_i represents the polynomial whose degree is $i-1$. These basis functions are obtained from the monomials 1, x, x^2, $\cdots$ by Gram-Schmidt orthonormalization over $[a,b]$ which is the interval appearing in the definition of $\mathcal{H}$.

Since any function in $\mathcal{H}$ is expressable as a linear combination of these basis functions, the function ϕ of the previous section should have the following expansion[5]

$$\phi(x) \equiv \sum_{i=1}^{\infty} \phi_i u_i(x) \tag{4}$$

where ϕ_i symbols stand for the scalars which can be determined uniquely when $\phi(x)$ is given. If we use this expansion in (2) then we can obtain

$$\mathcal{F}\left(\mathcal{L},\phi\right) \equiv \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i\phi_j \left\langle u_i \left|\mathcal{L}\left[\mathcal{I}-\mathcal{P}_{\phi}\right]\mathcal{L}\right| u_j\right\rangle \tag{5}$$

where the sum of squares of the complex modulii of ϕ scalars is 1 if the norm of $\phi(x)$ is 1, otherwise the right hand side of (5) must be divided by the sum of squares of the complex modulii of these ϕ_i's. The removal of the condition $\|\phi\| = 1$ brings more flexibility in the correct interpretation of what we will do in our coming investigations. Henceforth, we do not consider this condition and rewrite (5) as follows

$$\mathcal{F}\left(\mathcal{L},\phi\right) \equiv \frac{\boldsymbol{\phi}^{\dagger}\mathbf{F}\left(\mathcal{L},\phi\right)\boldsymbol{\phi}}{\boldsymbol{\phi}^{\dagger}\boldsymbol{\phi}} \tag{6}$$

where $\dagger$ symbol stands for the hermitian conjugation (complex conjugation plus transposition) and

$$\begin{aligned}
\boldsymbol{\phi}^{\dagger} &\equiv \left[\,\phi_1\, ...\, \phi_i\, ...\,\right] \\
\mathbf{F}\left(\mathcal{L},\phi\right) &\equiv \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & \cdots \\ \mathcal{F}_{21} & \mathcal{F}_{22} & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix} \\
\mathcal{F}_{ij} &\equiv \left\langle u_i \left|\mathcal{L}\left[\mathcal{I}-\mathcal{P}_{\phi}\right]\mathcal{L}\right| u_j\right\rangle, \\
&\quad 1 \le i,j < \infty
\end{aligned} \tag{7}$$

We can now recall the expectation value of $\mathcal{L}$ from (1) and rewrite it as follows when the condition $\|\phi\| = 1$ is not imposed.

$$\left\langle \phi \left|\mathcal{L}\right| \phi\right\rangle \equiv \frac{\boldsymbol{\phi}^{\dagger}\mathbf{L}\boldsymbol{\phi}}{\boldsymbol{\phi}^{\dagger}\boldsymbol{\phi}} \tag{8}$$

where $\mathbf{L}$ is the matrix representation of $\mathcal{L}$ over $\mathcal{U}$ in $\mathcal{H}$. $\mathbf{L}$ can also be called "Expectation Matrix of $\mathcal{L}$".[1]

(8) and its repetition for $\mathcal{L}^2$ enables us to join the last two equalities of (7) in a single formula as follows

$$\mathbf{F}\left(\mathcal{L},\phi\right) \equiv \frac{\boldsymbol{\phi}^{\dagger}\mathbf{L}\left[\mathbf{I}-\mathbf{P}_{\phi}\right]\mathbf{L}\boldsymbol{\phi}}{\boldsymbol{\phi}^{\dagger}\boldsymbol{\phi}} \tag{9}$$

where $\mathbf{I}$ stands for the unit infinite matrix and $\mathbf{P}_\phi$ is defined through the following equalities

$$\begin{aligned} \mathbf{P}_\phi &\equiv \overline{\phi}\,\overline{\phi}^\dagger \\ \overline{\phi} &\equiv \frac{\phi}{\left(\phi^\dagger \phi\right)^{\frac{1}{2}}} \end{aligned} \tag{10}$$

As can be seen from these equalities $\mathbf{P}_\phi$ is the matrix which projects any given vector from the infinite dimensional cartesian space to this space's subspace spanned by the vector ϕ.

The one–dimensionality of the abovementioned subspace is of course an undesired limitation which urges us to extend what we have done here to the case of more than one dimensional subspaces. To get more flexibility in our further investigations we can generalize the definition in the last definition of (7) as follows

$$\begin{aligned} \mathcal{F}_{ij} &\equiv \left\langle \phi_i \left| \mathcal{L} \left[\mathcal{I} - \mathcal{P}_\gamma \right] \mathcal{L} \right| \phi_j \right\rangle, \\ & \quad 1 \le i, j < \infty \end{aligned} \tag{11}$$

where $\mathcal{P}_\gamma$ denotes the operator which projects any given function in $\mathcal{H}$ to a subspace (with finite or infinite dimension, here we prefer infinite dimension for generality) spanned by the following set of orthonormal functions in $\mathcal{H}$

$$\Gamma \equiv \langle \gamma_i(x) \rangle_{i=1}^{\infty} \tag{12}$$

As can be noticed without remarkable difficulty the fluctuation matrix defined by (11) identically vanishes when Γ is a complete basis set for $\mathcal{H}$. Otherwise we get again a nonnegative matrix. The $\gamma_i(x)$ functions need not to be polynomials as the ones in $\mathcal{U}$. Now by having these entities we can denote the fluctuation matrix of $\mathcal{L}$ by $\mathbf{F}(\mathcal{L}, \mathcal{U}, \Gamma)$ and call "Fluctuation Matrix of the Operator $\mathcal{L}$ over the basis set $\mathcal{U}$ with respect to the subspace set Γ". In this sense the definitions given in (7) are the specification of this general definition as "Fluctuation Matrix of the Operator $\mathcal{L}$ over the basis set $\mathcal{U}$ with respect to the subspace set $\Gamma = \{\phi\}$".

Since both $\mathcal{U}$ and Γ are considered as infinite sets it is better to define the following truncated fluctuation matrices for practical reasons in applications.

$$\begin{aligned} \mathbf{F}(\mathcal{L}, \mathcal{U}, \Gamma, m, n) &\equiv \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & \cdots \\ \mathcal{F}_{21} & \mathcal{F}_{22} & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix} \\ \mathcal{F}_{ij} &\equiv \left\langle u_i \left| \mathcal{L} \left[\mathcal{I} - \mathcal{P}_\gamma^{(n)} \right] \mathcal{L} \right| u_j \right\rangle, \\ & \quad 1 \le i, j < m \\ \mathcal{P}_\gamma^{(n)} g(x) &\equiv \sum_{i=1}^{n} (\gamma_i, g)\, \gamma_i(x) \end{aligned} \tag{13}$$

where the independent variable dependence of the inner product arguments are not explicitly shown as usual since they are playing the role of the indices of the vector algebra and the function $g(x)$ stands for an arbitrary function in $\mathcal{H}$. We call $\mathbf{F}(\mathcal{L}, \mathcal{U}, \Gamma, m, n)$ "Fluctuation Matrix Truncation of $\mathcal{L}$ at the order m and n" where m and n mean that only first m and n numbers of the functions of $\mathcal{U}$ and Γ are taken into consideration. Although $\mathcal{U}$, Γ, m, and n are considered different for the sake of generality in the mathematical sense, most practical cases focuses on the case where $\mathcal{U} = \Gamma$ and $m = n$. Hence we devote this work on this focus in the coming sections.

3 Spectral Components of Fluctuation Matrices For Mostly Interested Cases

Let us now confine ourselves into the case where the fluctuation matrix of a function type multiplicative algebraic operator on the same set for basis and subspace sets with the same truncation level.

$$\begin{aligned} \mathcal{F}_{ij} &\equiv \left\langle u_i \left| f \left[\mathcal{I} - \mathcal{P}^{(n)} \right] f \right| u_j \right\rangle, \\ & \quad 1 \le i, j < n \\ \mathcal{P}^{(n)} g(x) &\equiv \sum_{i=1}^{n} (u_i, g)\, u_i(x), \\ \mathbf{F}(f, \mathcal{U}, n) &\equiv \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & \cdots \\ \mathcal{F}_{21} & \mathcal{F}_{22} & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix} \end{aligned} \tag{14}$$

The basis functions which are polynomials of x can also be defined through the following Rodriguez equation

$$\begin{aligned} u_i(x) &\equiv A_i \frac{d^{i-1}}{dx^{i-1}} \left[(b-x)^{i-1} (x-a)^{i-1} \right], \\ & \quad 1 \le i < \infty \end{aligned} \tag{15}$$

where the constant A_i is determined to make the norm of $u_i(x)$ 1 and is given through the following equality

$$A_i \equiv \frac{\sqrt{2i-1}(-1)^{i-1}}{(i-1)!(b-a)^{i-\frac{1}{2}}}, \quad 1 \le i < \infty \tag{16}$$

where the parity in the square rooting procedure is chosen in such a way that the highest power of $u_i(x)$ is positive for all i values.

As is expected these polynomials are mutually orthogonal over the interval $[\,a,b\,]$. Now the first equation in (14) can be rewritten as follows

$$\mathcal{F}_{ij} = \sum_{k=n+1}^{\infty} (u_i, f u_k)(u_k, f u_j), \quad 1 \le i,j \le n \tag{17}$$

We can also easily show that the following equality holds because of the orthogonality

$$(u_i, p u_k) = 0, \qquad n+1 \le k < \infty \tag{18}$$

as long as p is any polynomial whose degree is at most $k-i-1$ in powers of x. This urges us to choose $p(x)$ as the $k-i-1$. degree truncation of the Taylor series expansion of $f(x)$ around a point c in the interval $[\,a,b\,]$ as long as the convergence domain of this series includes the interval $[\,a,b\,]$. So the subtraction of such truncation polynomial from $f(x)$ in the inner product $(u_i, f u_k)$ does not change the value of inner product. We can define the following functions

$$\varphi_j(x) \equiv \sum_{i=j}^{\infty} \frac{1}{i!} f^{(i)}(c)(x-c)^i, \quad 0 \le j < \infty \tag{19}$$

and replace (17) with the following equality

$$\mathcal{F}_{ij} = \sum_{k=n+1}^{\infty} (u_i, \varphi_{n-i+1} u_k)(u_k, \varphi_{n-j+1} u_j), \quad 1 \le i,j \le n. \tag{20}$$

which implies

$$\mathcal{F}_{ij} \equiv \left\langle u_i \middle| \varphi_{n-i+1} \left[\mathcal{I} - \mathcal{P}^{(n)} \right] \varphi_{n-j+1} \middle| u_j \right\rangle, \quad 1 \le i,j < n. \tag{21}$$

Since one can easily show that

$$\left\| \varphi_{n-i+1} \left[\mathcal{I} - \mathcal{P}^{(n)} \right] \varphi_{n-j+1} \right\| \le \| \varphi_{n-i+1} \| \times \| \varphi_{n-j+1} \| \quad 1 \le i,j < n. \tag{22}$$

due to the fact that the norm of a projection operator or its complement to the Hilbert space under consideration is 1. The magnitudes of the elements of the fluctuation matrices in (14) decrease as one goes upward and/or leftward over the elements for a sufficiently smooth function $f(x)$ since $\varphi_i(x)$ tends to diminish when the index i increases as long as the series representation of $f(x)$ used in its definition converges. This implies that the eigenfunction of the above fluctuation matrix for its greatest eigenvalue will have its dominant element at n. position.

Now we can write the following equality by using (15) and (16) for an arbitrary function $g(x)$ in $\mathcal{H}$ which is the Hilbert space of the square integrable functions over $[\,a,b\,]$. $g(x)$ is chosen in such a way that its Taylor series' convergence domain includes the interval $[\,a,b\,]$.

$$\int_a^b dx g(x) u_i(x) = \frac{\sqrt{2i-1}}{(i-1)!}(b-a)^{\frac{1}{2}-i} \times \int_a^b dx (b-x)^{i-1}(x-a)^{i-1} g^{(i-1)}(x), \quad 1 \le i < \infty \tag{23}$$

where $(n-1)$ number of consecutive integrations by parts are used. If we define a new integration variable via $y \equiv (x-a)/(b-a)$ then we can write

$$\int_a^b dx g(x) u_i(x) = \frac{\sqrt{2i-1}}{(i-1)!}(b-a)^{i-\frac{1}{2}} \times \int_0^1 dy y^{i-1}(1-y)^{i-1} \times g^{(i-1)}\left(\frac{b+a}{2} + (b-a)\left(y - \frac{1}{2} \right) \right), \quad 1 \le i < \infty \tag{24}$$

which implies

$$\lim_{b \to a} \int_a^b dx g(x) u_i(x) = \frac{(b-a)^{i-\frac{1}{2}}}{\sqrt{2i-1}} \frac{(i-1)!}{(2i-2)!} g^{(i-1)}\left(\frac{b+a}{2} \right), \quad 1 \le i < \infty \tag{25}$$

where we have used the continuity of $g(x)$ in $[\,a,b\,]$ and the following formula through the definition of beta function.

$$\int_0^1 dy y^{i-1}(1-y)^{i-1} = \frac{(i-1)!(i-1)!}{(2i-2)!}, \quad 0 \le i < \infty \tag{26}$$

Let us now go back to (20) where p polynomial was chosen pessimistically k-independent. Its more optimistic form is as follows

$$\mathcal{F}_{ij} = \sum_{k=n+1}^{\infty} (u_i, \varphi_{k-i+1} u_k)(u_k, \varphi_{k-j+1} u_j), \quad 1 \le i,j \le n. \tag{27}$$

which urges us to concentrate on the inner products in the summand. We can write the following equation by using (25) and integral definition of the inner product

$$\lim_{b\to a}(u_i,\varphi_{k-i}u_k) = \frac{(b-a)^{k-\frac{1}{2}}}{\sqrt{2k-1}}\frac{(k-1)!}{(2k-2)!} \times (u_i(x)\varphi_{k-i}(x))^{(k-1)}\Big|_{x=\frac{a+b}{2}}, \quad 1\le i\le n,\ \ n+1\le k<\infty \qquad (28)$$

where the superscript between parantheses stands for the order of the differentiation.

A careful investigation shows that

$$\lim_{b\to a}u_i(x) = \sqrt{2i-1}\binom{2i-2}{i-1}(b-a)^{\frac{1}{2}-i} \times\left(x-\frac{a+b}{2}\right)^{i-1},\quad 1\le i<\infty \qquad (29)$$

where the second factor at the right hand side stands for the binomial coefficient. The result in the last equality enables us to write the following formula after certain intermediate steps

$$(u_i(x)\varphi_{k-i}(x))^{(k-1)}\Big|_{x=\frac{a+b}{2}} = \frac{\sqrt{2i-1}}{(k-i)!} \times\binom{2i-2}{i-1}(b-a)^{\frac{1}{2}-i}f^{(k-i)}\left(\frac{a+b}{2}\right), \quad 1\le i\le n,\ \ n+1\le k<\infty \qquad (30)$$

This implies

$$\lim_{b\to a}(u_i,\varphi_{k-i}u_k) = \frac{(b-a)^{k-i}(k-1)!}{(k-i)!(2k-2)!} \times\frac{\sqrt{2i-1}}{\sqrt{2k-1}}\binom{2i-2}{i-1}f^{(k-i)}\left(\frac{a+b}{2}\right), \quad 1\le i\le n,\ \ n+1\le k<\infty \qquad (31)$$

(31) and its companion where i is replaced by j can be used in (27) and all terms of the resulting infinite sum but the most dominating one when b approaches a are ignored to get

$$\lim_{b\to a}\mathcal{F}_{ij} = \overline{\mathcal{F}}_i\overline{\mathcal{F}}_j, \qquad 1\le i,j\le n \qquad (32)$$

where

$$\lim_{b\to a}\overline{\mathcal{F}}_i = \frac{(b-a)^{n-i+1}n!}{(n-i+1)!(2n)!}\frac{\sqrt{2i-1}}{\sqrt{2n+1}} \times\binom{2i-2}{i-1}f^{(n-i+1)}\left(\frac{a+b}{2}\right), \quad 1\le i\le n \qquad (33)$$

(32) means that the fluctuation matrix under consideration becomes an outer product at the zero length interval limit. Hence it has just a single positive eigenvalue and all other eigenvalues vanish. The eigenvector of the nonzero eigenvalue is parallel to the vector whose elements are $\overline{\mathcal{F}}_i$'s. As long as f is sufficiently smooth one can decide that the element of this eigenfunction with the dominating magnitude is its last element, that is, $\overline{\mathcal{F}}_n$. Therefore we can expect that this eigenvector approaches $\mathbf{e}_n$, n–th cartesian unit vector whose all elements vanish except the n–th one as n grows unboundedly. We are not going to intend to prove this conjecture and certain similar ones in rigorous mathematical manipulations. Instead we suffice to give numerical results for certain univariate functions as the integration interval length in the definition of the fluctuation matrix diminishes to zero in the coming section.

4 Illustrative Implementations

We experimented the theory given here for certain univariate functions. In each numerical implementations we have chosen a specific dimension and then evaluated the eigenvalues and the eigenfunctions of the fluctuation matrix for various interval length values. The values are chosen for a truncation dimension n such that the angle between the i–th eigenvector and the i–th cartesian unit vector approaches zero as we trace eigenvalues in ascending ordering. We give the situation for the function given by

$$f(x)\equiv\sqrt{1+x} \qquad (34)$$

only for exemplification. This function has a branch point located at $x=-1$ and infinity, and hence, it converges everywhere in the disc centered at $x=\frac{1}{2}$ with the radius $\frac{3}{2}$. For $n=3$ the smallest eigenvalue is given for certain descendingly ordered half interval length values (hilv)

$hilv$	$eigenvalue$
0.500	1.72×10^{-13}
0.100	1.47×10^{-20}
0.050	1.42×10^{-23}
0.010	1.45×10^{-30}
0.005	1.42×10^{-33}
0.001	1.45×10^{-40}

Same kind of tables for the second and third eigenvalues are also constructed, although we do not give them similar behavior of above tables are observed. In all tables the first fractional digit contains the rounded contributions (if any) from the less important digits.

The values of the angle between the eigenfunction of the smallest eigenvalue and the first cartesian unit vector versus descendingly ordered half integral length values are given in the following table

$hilv$	$angle$
0.500	1.45×10^{-1}
0.100	2.89×10^{-2}
0.050	1.44×10^{-2}
0.010	2.89×10^{-3}
0.005	1.44×10^{-3}
0.001	2.89×10^{-4}

Same kind of tables for the angles between the other eigenvectors and corresponding cartesian unit vectors are also constructed although we do not give them here. In all tables the second fractional digit contains the rounded contributions (if any) from the less important digits.

Similar investigations can be made for different kind of functions. Although we do not give here explicitly similar tables to above ones for some other kind of functions like $ln(1+x)$, $sin(x)$, $\sqrt{\frac{1}{1-x}}$ are constructed and support our conjecture which states that in the limiting form the j–th eigenfunction approaches j–th cartesian unit vector in ascending ordering of eigenvalues.

5 Conclusion

In this work we have focused on fluctuation matrix behavior when the interval used in its definition tends to vanish. Our observations show that the eigenfunctions of the fluctuation matrix tends to go to the corresponding cartesian unit vector (j–th to j–th, $j = 1, ..., n$) when the eigenvalues are ordered from the smallest to greatest when the interval length diminishes to zero. This is observed for the functions whose convergence domain contains the interval under consideration. The observations support what we have conjectured here. We have not given a rigorous proof of our conjecture. However, the theory presents quite mild mathematical structure to prove them. This is an important and rich area of research. The relation between singularities of the core function in the fluctuation matrix and the eigenfunction behavior for diminishing interval lengths may be interesting for future works. Certain approximation methods can be developed for the evaluation of the eigenpairs even when the interval length is far from the zero limit.

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