Boundary estimates for solutions of non-homogeneous boundary value problems on graphs

SONJA CURRIE
University of the Witwatersrand
School of Mathematics
Private bag 3
P.O. Wits 2050
South Africa

BRUCE A. WATSON
University of the Witwatersrand
School of Mathematics
Private bag 3
P.O. Wits 2050
South Africa

Abstract: We consider the Sturm-Liouville operator on a graph and give bounds for the norms of the boundary values of solutions to the non-homogeneous boundary value problem in terms of the norm of the non-homogeneity. In addition the eigenparameter dependence of these bounds is studied.

Key–Words: Differential Operators on Graphs, Boundary Estimates

1 Introduction

We consider the Sturm-Liouville equation

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \qquad (1)$$

where q is real-valued and essentially bounded (with respect to Lebesgue measure), on a weighted graph G with formally self-adjoint boundary conditions at the nodes. For characterisations of self-adjoint boundary value problems on graphs and associated boundary conditions we refer the reader to [5] and [11].

In [10], it was shown that the geometry of a non-commensurate simple graph is uniquely dependent on the spectrum of the Laplacian on the graph. I.e. for zero potential they reconstructed the boundary conditions (of a specific type) from a single spectrum. In [6] spectral asymptotics were given for l on graphs where all edges are of equal length while in [7] and [8] eigenvalue asymptotics were given for l on general compact graphs via matrix Prüfer angle techniques and Dirichlet-Neumann bracketing respectively. Variational aspects of boundary value problems on graphs were studied in [2], [8] and [20], and on trees in [19]. Sturmian oscillation theory was extended to Sturm-Liouville operators on graphs by Pokornyi and Pryadiev, and Pokornyi, Pryadiev and Al-Obeid, in [15] and [16].

Sturm-Liouville problems on finite graphs arise naturally in quantum mechanics and circuit theory, [3, 9]. In [13] and the bibliography thereof one can find an extensive collection of physical systems that give rise to Sturm-Liouville problems on graphs.

Here we consider solutions of non-homogeneous Sturm-Liouville problems on graphs and their a priori estimates. Particular attention is given to the relationship between the norm of the solutions to the non-homogeneous boundary value problem on the boundary of the graph and the norm of the non-homogeneous term on the graph, see Corollary 3.2. In addition the eigenparameter dependence of this relationship is explored. To complete the paper an example is provided in Section 4, illustrating Corollary 3.2. The results obtained in this paper rely on an ability to make the transition between local results (on each edge) and global results (on the whole graph). Thus the method employed involves two main steps: establishing a local result on each edge; and the transition from the local results to a global result on the whole graph.

It should be noted that for Sturmian systems it is only possible to find boundary estimates of the form given in this paper for two specific types of systems: firstly for non-weighted systems with general, essentially bounded, Hermitian potential (not necessarily diagonal); secondly, for diagonal systems, which are equivalent to considering Sturm-Liouville equations on graphs as is done in this paper, see [7] for the equivalence.

In [17, 18], Schauder considers interior estimates and estimates near the boundary for solutions of second order elliptic boundary value problems. His estimates near the boundary are for solutions of the Dirichlet problem. Estimates near the boundary for other than Dirichlet boundary conditions have been obtained by Miranda, [14], for second order elliptic boundary value problems and by Agmon, Douglis, Nirenberg and Browder, [1, 4], for arbitrary order elliptic operators.

In the above references it should be noted that the estimates are given in a region near the boundary whereas our results provide estimates on the boundary.

2 Preliminaries

Let G denote a directed graph with a finite number of nodes and edges, with each edge parametrized by path-length and having finite length. Each edge, e_i, of length say l_i can thus be considered as the interval $[0, l_i]$. Having made this identification, it is possible to consider the differential equation (1) on the graph G to be the collection of differential equations

$$-\frac{d^2y_i}{dx^2} + q_i(x)y_i = \lambda y_i, \quad x \in [0, l_i], \tag{2}$$

for $i = 1, ..., K$, where q_i and y_i denote the restrictions of q and y to e_i, respectively.

It is now possible, at each node, ν, to specify boundary conditions in terms of the values of y and y' at ν on each of the incident edges. In particular if the edges which originate at node ν are $e_i, i \in \Lambda_s(\nu)$ and the edges which terminate at node ν are $e_i, i \in \Lambda_e(\nu)$ then the boundary conditions at ν are of the form

$$\sum_{j\in\Lambda_s(\nu)} \left[\alpha_{ij}y_j(0) + \beta_{ij}{y'}_j(0)\right] + \sum_{j\in\Lambda_e(\nu)} \left[\gamma_{ij}y_j(l_j) + \delta_{ij}{y'}_j(l_j)\right] = 0, \tag{3}$$

for $i = 1, ..., N(\nu)$ where $N(\nu)$ is the number of linearly independent boundary conditions at node ν.

Remark It should be noted that by setting $\alpha_{ij} = 0 = \beta_{ij}$ for $i = 1, ..., N(\nu)$ with $j \notin \Lambda_s(\nu)$ and $\gamma_{ij} = 0 = \delta_{ij}$ for $i = 1, ..., N(\nu)$ with $j \notin \Lambda_e(\nu)$, after relabelling the condition (3), taking all nodes into account, may be written as

$$\sum_{j=1}^{K} \left[\alpha_{ij}y_j(0) + \beta_{ij}{y'}_j(0)\right] + \sum_{j=1}^{K} \left[\gamma_{ij}y_j(l_j) + \delta_{ij}{y'}_j(l_j)\right] = 0, \tag{4}$$

for $i = 1, ..., N$, where N is the total number of linearly independent boundary conditions.

Define $\mathcal{L}^2(G)$ to be the Hilbert space of all $f : G \to C$ with $f_i \in \mathcal{L}^2(0, l_i)$. Here the inner product on $\mathcal{L}^2(G)$ is given by

$$(f, g) = \sum_{i=1}^{K} \int_0^{l_i} f|_{e_i}\bar{g}|_{e_i}\, dt.$$

The above boundary value problem on G can be reformulated as an operator eigenvalue problem, [5], by setting

$$Lf = -f'' + qf$$

with domain
$\mathcal{D}(L)$
$= \{f \mid f, f' \in AC, l(f) \in \mathcal{L}^2(G),\ f$ obeying (3) $\}$.

In this setting, the formal self-adjointness of (2)-(3) ensures that the operator L on $\mathcal{L}^2(G)$ is a self-adjoint operator, see [21, p. 77-78].

3 Boundary Estimates

Theorem 3.1 *Let $\lambda = -k^2$, $k > 0$, then for y a solution of the boundary value problem (2)-(3),*

$$||y||_{\mathcal{L}^2(G)} = \frac{1}{\sqrt{2k}}||y|_{\partial G}||_{\mathcal{L}^2(\partial G)} \left(1 + O\left(\frac{1}{k}\right)\right) \tag{5}$$

as $k \to \infty$, where ∂G denotes the boundary of G.

Proof: Consider the second order Sturm-Liouville problem on the interval $[0, l_i]$ given by

$$-y_i'' + qy_i = \lambda y_i \tag{6}$$

with non-homogeneous Dirichlet boundary conditions

$$y_i(0) = \alpha_i(k) \quad \text{and} \quad y_i(l_i) = \beta_i(k). \tag{7}$$

Let λ_0^i denote the least eigenvalue of (6) on $(0, l_i)$ with Dirichlet boundary conditions, $y_i(0) = 0 = y_i(l_i)$. Taking $\lambda < \Lambda := \min_{i=1,...,K} \lambda_0^i$ we have that (6)-(7) has a unique solution for each $\alpha_i(k)$, $\beta_i(k)$ and each $i = 1, \ldots, K$.

From [12, Appendix A1], the fundamental solutions of (6) obeying the boundary conditions

$$u_1(0) = \quad 1 \quad = u_2'(0), \tag{8}$$
$$u_1'(0) = \quad 0 \quad = u_2(0) \tag{9}$$

are given asymptotically for large $k > 0$, by

$$u_1(t) = \cosh kt + O\left(\frac{e^{kt}}{k}\right), \tag{10}$$

$$u_2(t) = \frac{1}{k}\sinh kt + O\left(\frac{e^{kt}}{k^2}\right) \tag{11}$$

with corresponding derivatives

$$u_1'(t) = k\sinh kt + O(e^{kt}), \tag{12}$$
$$u_2'(t) = \cosh kt + O\left(\frac{e^{kt}}{k}\right), \tag{13}$$

uniformly with respect to t.

Note that the Wronskian of $u_1(t)$ and $u_2(t)$ is equal to 1 for all t, i.e.

$$u_1(t)u_2'(t) - u_2(t)u_1'(t) = 1, \quad \text{for all } t.$$

From (7), (8) and (9)

$$\begin{aligned} y_i(t) &= \alpha_i(k)u_1(t) - \frac{u_1(l_i)\alpha_i(k) - \beta_i(k)}{u_2(l_i)}u_2(t) \\ &= \frac{[-\alpha_i(k)(-u_1(t)u_2(l_i) + u_1(l_i)u_2(t))]}{u_2(l_i)} \\ &+ \frac{\beta_i(k)u_2(t)}{u_2(l_i)}. \end{aligned} \tag{14}$$

Let

$$w(t) := -u_1(t)u_2(l_i) + u_1(l_i)u_2(t),$$

then w is the solution of (6) with

$$\begin{aligned} w(l_i) &= 0 \\ w'(l_i) &= -u_1'(t)u_2(l_i) + u_1(l_i)u_2'(t) = 1. \end{aligned}$$

Thus from [12, Appendix A1], for large $k > 0$,

$$w(t) = \frac{1}{k}\sinh k(t - l_i) + O\left(\frac{e^{k(l_i-t)}}{k^2}\right), \tag{15}$$

uniformly in t.

Now

$$u_2^2(l_i) = \frac{e^{2kl_i}}{4k^2}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{16}$$

Substituting (11) and (15) into (14) we obtain

$$\begin{aligned} &y_i(t) \\ &= \frac{1}{u_2(l_i)}[\beta_i(k)u_2(t) - \alpha_i(k)w(t)] \\ &= \frac{1}{ku_2(l_i)}\left[\beta_i(k)\left(\sinh kt + O\left(\frac{e^{kt}}{k}\right)\right) + \right. \\ &\left. \alpha_i(k)\left(\sinh k(l_i - t) + O\left(\frac{e^{k(l_i-t)}}{k}\right)\right)\right]. \end{aligned}$$

Squaring this gives

$$\begin{aligned} &y_i^2(t) \\ &= \frac{1}{k^2u_2^2(l_i)}\left[\beta_i^2(k)\left(\sinh kt + O\left(\frac{e^{kt}}{k}\right)\right)^2\right. \\ &+ 2\alpha_i(k)\beta_i(k)\left(\sinh kt + O\left(\frac{e^{kt}}{k}\right)\right) \times \\ &\left(\sinh k(l_i - t) + O\left(\frac{e^{k(l_i-t)}}{k}\right)\right) \\ &+ \left.\alpha_i^2(k)\left(\sinh k(l_i - t) + O\left(\frac{e^{k(l_i-t)}}{k}\right)\right)^2\right]. \end{aligned}$$

By the Schwartz inequality

$$\alpha_i^2(k) + \beta_i^2(k) \geq |2\alpha_i(k)\beta_i(k)|,$$

hence for large $k > 0$, $y_i^2(t)/(\alpha_i^2(k) + \beta_i^2(k))$ is bounded on $(0, l_i)$. Thus Lebesgue's dominated convergence theorem may be applied, and it suffices to consider the pointwise limit of $y_i^2(t)/(\alpha_i^2(k)+\beta_i^2(k))$ for $t \in (0, l_i)$. For $t \in (0, l_i)$ and $k \to \infty$,

$$\begin{aligned} y_i^2(t) &= \frac{1}{k^2u_2^2(l_i)}\left[\frac{\beta_i^2(k)e^{2kt}}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\right. \\ &+ \frac{2\alpha_i(k)\beta_i(k)e^{kl_i}}{4}\left(1 + O\left(\frac{1}{k}\right)\right) \\ &+ \left.\frac{\alpha_i^2(k)e^{2k(l_i-t)}}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\right]. \end{aligned}$$

Integrating from 0 to l_i gives

$$\begin{aligned} &\textstyle\int_0^{l_i} y_i^2(t)\,dt \\ &= \frac{1}{k^2u_2^2(l_i)}\left[\frac{\beta_i^2(k)}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\frac{e^{2kl_i} - 1}{2k}\right. \\ &+ \frac{2l_i\alpha_i(k)\beta_i(k)e^{kl_i}}{4}\left(1 + O\left(\frac{1}{k}\right)\right) \\ &+ \left.\frac{\alpha_i^2(k)}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\frac{e^{2kl_i} - 1}{2k}\right] \\ &= \frac{e^{2kl_i}}{8k^3u_2^2(l_i)}\left[\left(1 + O\left(\frac{1}{k}\right)\right) \times\right. \\ &\left.\left(\beta_i^2(k) + \alpha_i^2(k) + 4\alpha_i(k)\beta_i(k)kl_ie^{-kl_i}\right)\right] \\ &= \frac{e^{2kl_i}}{8k^3u_2^2(l_i)}(\alpha_i^2 + \beta_i^2)\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned}$$

Using (16) gives

$$\begin{aligned} \int_0^{l_i} y_i^2(t)\,dt &= ||y_i||^2_{\mathcal{L}^2(0,l_i)} \\ &= \frac{\alpha_i^2 + \beta_i^2}{2k}\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned}$$

Therefore

$$||y_i||^2_{\mathcal{L}^2(0,l_i)} = \frac{||y_i|_{\partial(0,l_i)}||^2_{\mathcal{L}^2(\partial(0,l_i))}}{2k}\left(1+O\left(\frac{1}{k}\right)\right).$$

Summing over $i = 1,\ldots,K$ proves the theorem. ■

The following corollary gives bounds for the boundary norm of solutions to the non-homogeneous boundary value problem in terms of the non-homogeneous term.

Corollary 3.2 *There exists a constant $C > 0$ such that, for $k > 0$ large,*

$$\frac{C}{k^{\frac{3}{2}}}||f||_{\mathcal{L}^2(G)} \geq ||y|_{\partial G}||_{\mathcal{L}^2(\partial G)}, \tag{17}$$

for all $f \in \mathcal{L}^2(G)$, where y is the solution of

$$-y'' + qy = \lambda y + f, \tag{18}$$

obeying the boundary conditions (3), for $\lambda = -k^2$.

Proof: Let G_λ denote the Green's operator of the boundary value problem (2)-(3) and let G^D_λ denote the Green's operator of the boundary value problem (2) but with Dirichlet boundary conditions at every node (i.e. y is zero at all nodes).

We note that

$$(l-\lambda)(G_\lambda - G^D_\lambda)f = f - f = 0$$

for $f \in \mathcal{L}^2(G)$ and where l is as given in (1). Thus $(G_\lambda - G^D_\lambda)f$ is a solution of (2) and from Theorem 3.1 obeys (5) hence we obtain that, since $(1+O(\frac{1}{k})) \geq \frac{1}{\sqrt{2}}$ for large k,

$$\begin{aligned}&||(G_\lambda - G^D_\lambda)f||_{\mathcal{L}^2(G)} \geq \\ &\frac{1}{2\sqrt{k}}||[(G_\lambda - G^D_\lambda)f]|_{\partial G}||_{\mathcal{L}^2(\partial G)},\end{aligned} \tag{19}$$

for all $f \in \mathcal{L}^2(G)$. But

$$[(G_\lambda - G^D_\lambda)f]|_{\partial G} = [G_\lambda f]|_{\partial G}$$

giving

$$||(G_\lambda - G^D_\lambda)f||_{\mathcal{L}^2(G)} \geq \frac{1}{2\sqrt{k}}||[G_\lambda f]|_{\partial G}||_{\mathcal{L}^2(\partial G)}.$$

Now as G_λ and G^D_λ are both resolvent operators we have

$$||(G_\lambda - G^D_\lambda)f||_{\mathcal{L}^2(G)} \leq C\frac{||f||_{\mathcal{L}^2(G)}}{|\lambda|} = C\frac{||f||_{\mathcal{L}^2(G)}}{k^2} \tag{20}$$

for $\lambda \to -\infty$, where $C > 0$ is a constant.

Hence, combining (19) and (20), we obtain that

$$\begin{aligned} 2C\frac{||f||_{\mathcal{L}^2(G)}}{k^2} &\geq \frac{1}{\sqrt{k}}||[(G_\lambda - G^D_\lambda)f]|_{\partial G}||_{\mathcal{L}^2(\partial G)} \\ &= \frac{1}{\sqrt{k}}||[G_\lambda f]|_{\partial G}||_{\mathcal{L}^2(\partial G)}.\end{aligned}$$

Taking $y = G_\lambda f$ gives (17). ■

Remark For the system

$$-Y'' + QY = \lambda WY + F, \tag{21}$$

with general self-adjoint boundary conditions of the form

$$AY(0) + BY'(0) + CY(1) + DY'(1) = 0, \tag{22}$$

for A, B, C and D constant matrices, where either
(i) $Q \in \mathcal{L}^\infty(0,1)$ is Hermitian (not necessarily diagonal), $W = I$ and $F \in \mathcal{L}^2(0,1)$ is Hermitian, i.e. a non-weighted system with general, Hermitian, $\mathcal{L}^\infty$ potential or
(ii) $Q \in \mathcal{L}^\infty(0,1)$ is real valued and diagonal, W is constant, real valued and diagonal and $F \in \mathcal{L}^2(0,1)$ is diagonal, i.e. a Sturm-Liouville boundary value problem on a graph, see [7],
the following result, corresponding to the above corollary, is obtained.

There exists a constant $C > 0$ such that for $k > 0$ large,

$$\frac{C}{k^{\frac{3}{2}}}||F||_{\mathcal{L}^2(0,1)} \geq ||\underline{Y}|_{\partial(0,1)}||_{\mathcal{L}^2(\partial(0,1))} \tag{23}$$

for $\underline{Y}$ the solution of (21), with $\lambda = -k^2$, obeying the boundary conditions (22).

4 Example

In this section we provide an example to illustrate Corollary 3.2. We also show that (17) is the best possible estimate that can be obtained.

Consider the second order differential equation

$$-y'' - \lambda y = e^{\alpha t} \tag{24}$$

on $[0,\pi]$ where α is a constant, i.e. a graph with a single edge of length π, with the boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y'(\pi) &= 0. \end{aligned} \tag{25}$$

Then the solutions of (24) on the interval $[0, \pi]$ are of the form

$$y = \frac{e^{\alpha t}}{k^2 - \alpha^2} + ae^{kt} + be^{-kt}, \qquad (26)$$

where a and b are constants, and $\lambda = -k^2$.

From (26) and the boundary conditions (25), the constants a and b are given as follows

$$\begin{aligned} a &= -\frac{\alpha e^{\pi\alpha} + ke^{-\pi k}}{k(k^2 - \alpha^2)(e^{k\pi} + e^{-k\pi})}, \\ b &= \frac{\alpha e^{\pi\alpha} - ke^{\pi k}}{k(k^2 - \alpha^2)(e^{k\pi} + e^{-k\pi})}. \end{aligned}$$

Substituting the constants back into (26) and evaluating at 0 and π gives

$$\begin{aligned} y(0) &= 0, \\ y(\pi) &= \frac{(k-\alpha)e^{\pi(k+\alpha)} + (k+\alpha)e^{\pi(\alpha-k)} - 2k}{k(k^2 - \alpha^2)(e^{k\pi} + e^{-k\pi})}. \end{aligned}$$

We now look at the case of $\alpha = 2k$. Then

$$y(\pi) = \frac{ke^{3\pi k} - 3ke^{\pi k} + 2k}{3k^3(e^{k\pi} + e^{-k\pi})} \simeq \frac{e^{2\pi k}}{3k^2}$$

giving

$$||y|_{\partial G}||_{\mathcal{L}^2(\partial G)} \simeq \frac{e^{2\pi k}}{3k^2}.$$

Also

$$||f||_{\mathcal{L}^2(G)} = \sqrt{\frac{e^{4\pi k} - 1}{4k}} \simeq \frac{e^{2\pi k}}{2\sqrt{k}},$$

and therefore

$$\frac{||f||_{\mathcal{L}^2(G)}}{||y|_{\partial G}||_{\mathcal{L}^2(\partial G)}} \simeq \frac{3}{2}k^{\frac{3}{2}}.$$

Thus

$$\frac{3}{2k^{\frac{3}{2}}}||f||_{\mathcal{L}^2(G)} \geq ||y|_{\partial G}||_{\mathcal{L}^2(\partial G)}.$$

Showing the power term in k to be optimal.

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