Implications of a Scale Invariant Model of Statistical Mechanics to Nonstandard Analysis and the Wave Equation

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Abstract: - A scale-invariant model of statistical mechanics is applied to examine the physical foundation of nonstandard analysis and to identify the nature and the range $(0_\beta,\ 0_{\beta-1})$ of nonstandard numbers and to establish the existence of infinitesimals $(0_\beta > L_{\infty\beta-2} > x_\beta > 0_{\beta-2})$. An invariant logarithmic definition of coordinate is presented and the concept of "measureless" or "dimensionless" numbers $(L'_{\infty\beta}, \lambda_\beta, 0_\beta) =$ $(L_{\infty\beta}, 1_\beta, 0_\beta)$ is described. Also, a scale-invariant definition of fractal dimension is introduced that suggest exceedingly large values 10^7 of fractal dimension. A scale invariant form of the wave equation is derived that applies to electromagnetic waves that propagate at the speed of light $v_t = c$ and gravitational waves that propagate at superluminal speeds $v_g \geq 2\times 10^{10}c$.

Key-Words: - Nonstandard analysis; Infinitesimals; Gravitational waves; Gravitational radiation.

1 Introduction

The universality of turbulent phenomena from stochastic quantum fields [1-16] to classical hydrodynamic fields [17-26] resulted in recent introduction of a scale-invariant model of statistical mechanics and its application to the field of thermodynamics [27-28]. The invariant forms of conservation equations were subsequently employed to present a modified theory of laminar flames [29].

In the present study, the invariant model of statistical mechanics is used to examine the physical foundation of nonstandard analysis and the existence of infinitesimals. An invariant logarithmic definition of coordinate is introduced and the concept of "measureless" or "dimensionless" numbers is described. Also, a scale-invariant definition of fractal dimension is presented in harmony with the classical definition. The invariant form of the equation of motion is applied to derive an invariant form of the wave equation. The implications of the results to "wave" versus "radiation" speeds for acoustic waves that propagate at speed of sound, electromagnetic waves that propagate at the speed of light $v_t = c$, as well as gravitational waves that propagates at superluminal speeds $v_g \geq 2\times 10^{10}c$ are also discussed.

2 Invariant Forms of the Conservation Equations for Reactive Fields

Following the classical methods [30-34], the invariant definitions of the density ρ_β, and the velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [28]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \qquad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \qquad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \tag{2}$$

The scale-invariant model of statistical mechanics from cosmic to tachyon scales is shown in Fig.1 A more detailed diagram for the intermediate scales of eddy-, cluster-, and molecular-dynamic is shown in Fig.2. Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \qquad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta \tag{3}$$

such that

$$\mathbf{V}_\beta = \mathbf{V}'_{\beta+1} \tag{4}$$

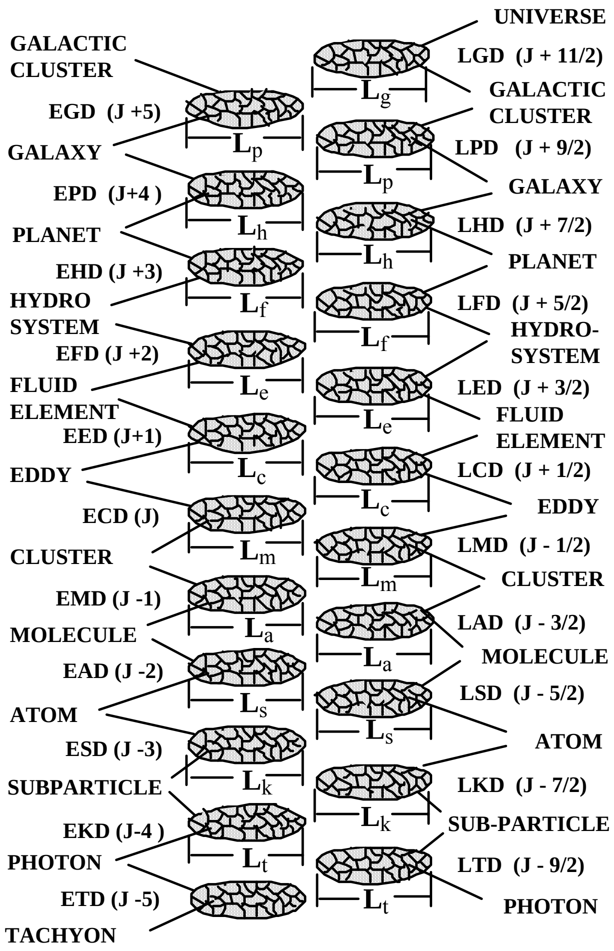

Fig.1. A scale-invariant view of statistical mechanics from cosmic to tachyon scales.

Following the classical methods [30-34], the scale-invariant forms of mass, thermal energy and momentum conservation equations at scale β are given as [29]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla.\left(\rho_\beta \mathbf{v}_\beta\right) = \Omega_\beta \tag{4}$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla.\left(\varepsilon_\beta \mathbf{v}_\beta\right) = 0 \tag{5}$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla.\left(\mathbf{p}_\beta \mathbf{v}_\beta\right) = -\nabla.\mathbf{P}_\beta \tag{6}$$

involving the *volumetric density* of thermal energy $\varepsilon_\beta = \rho_\beta h_\beta$ and linear momentum $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$. Also, Ω_β is the chemical reaction rate, h_β is the absolute enthalpy [27], and $\mathbf{P}_\beta$ is the partial stress tensor [30]

$$\mathbf{P}_\beta = m_\beta \int (\mathbf{u}_\beta - \mathbf{v}_\beta)(\mathbf{u}_\beta - \mathbf{v}_\beta) f_\beta du_\beta \tag{7}$$

In the derivation of (6) we have used the definition of the peculiar velocity (3) along with the identity

$$\overline{\mathbf{V}'_{\beta i}\mathbf{V}'_{\beta j}} = \overline{(\mathbf{u}_{\beta i} - \mathbf{v}_{\beta i})(\mathbf{u}_{\beta j} - \mathbf{v}_{\beta j})} = \overline{\mathbf{u}_{\beta i}\mathbf{u}_{\beta j}} - \mathbf{v}_{\beta i}\mathbf{v}_{\beta j} \tag{8}$$

The transport of mass, linear momentum, and thermal energy are considered to occur by both convection and diffusion. Hence, the local velocity $\mathbf{v}_\beta$ in (4)-(6) is expressed in terms of the convective $\mathbf{w}_\beta$ and the diffusive $\mathbf{V}_\beta$ velocities [29]

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \tag{9a}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \tag{9b}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_\beta \nabla \ln(\mathbf{p}_\beta) \tag{9c}$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$) are respectively the diffusive, the thermo-diffusive, the linear hydro-diffusive velocities.

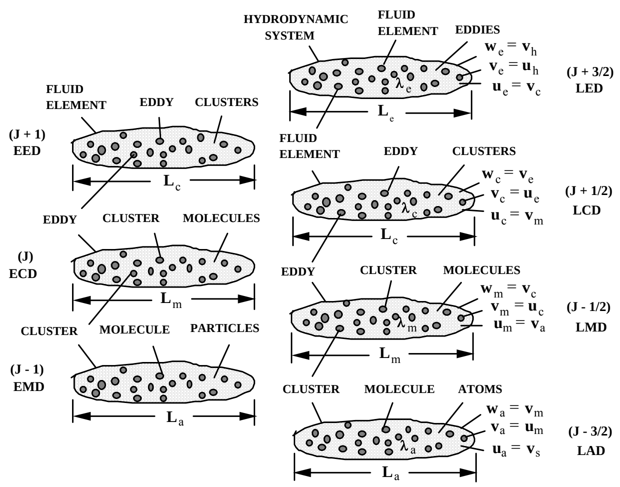

Fig.2 Hierarchy of statistical fields for equilibrium eddy-, cluster-, and molecular-dynamic scales and the associated laminar flow fields.

By substitutions from (9) in (4)-(6) and neglecting the cross-diffusion terms one obtains for constant transport coefficients the scale-invariant forms of conservation equations [29].

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta.\nabla\rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{10}$$

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta.\nabla T_\beta - \alpha_\beta \nabla^2 T_\beta = -\frac{h_\beta \Omega_\beta}{\rho_\beta c_{p\beta}} \tag{11}$$

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta.\nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta = -\frac{\nabla p_\beta}{\rho_\beta} - \frac{\mathbf{v}_\beta \Omega_\beta}{\rho_\beta} \tag{12}$$

3 Invariant Definitions of System, Element, and Atomic Lengths

In view of the model described in Fig.1, the spatial coordinate for the statistical field at scale β will be defined as [27]

$$x'_\beta = \ln N_{A\beta} \tag{13}$$

Therefore, the spatial distance of each statistical field is measured on the basis of the number of "atoms" of that particular statistical field $N_{A\beta}$. With definition (13) the counting of numbers must begin with the number zero naturally since it corresponds to one atom. The characteristic lengths of the (system, element, atom) = $(L_\beta, \lambda_\beta, \ell_\beta)$ at scale β are defined as

$$\begin{aligned} L'_{\infty\beta} &= \ln N_{AS\beta} \\ \lambda_\beta &= \ln N_{AE\beta} \\ \ell_\beta &= 0_\beta \quad , \quad N_{A\beta} = 1 \end{aligned} \tag{14}$$

where $(N_{AS\beta}, N_{AE\beta})$ respectively refer to the number of atoms in the system and the element. In view of the definitions in (13)-(14), the following schematic representation of the hierarchy of length scales may be obtained

$$\begin{array}{l} \bullet \quad \bullet \quad \bullet \\ \quad L'_{\infty\beta+1} \underline{\hspace{6em}} \lambda_{\beta+1} \underline{\hspace{3em}} 0_{\beta+1} \\ \hspace{12em} L'_{\infty\beta} \underline{\hspace{3em}} \lambda_\beta \underline{\hspace{1.5em}} 0_\beta \\ \hspace{22em} \bullet \quad \bullet \quad \bullet \end{array} \tag{15}$$

The size of the element and atom of adjacent statistical fields within the cascade (Fig.1) will be related by

$$N_{AE\beta} = N_{AS\beta-1} \tag{16}$$

$$N_{A\beta} = N_{AE\beta-1} \tag{17}$$

As a result of (14) and (16)-(17), system, element, and zero lengths of the adjacent statistical fields are related to each other as

$$\begin{aligned} \lambda_\beta &= L'_{\infty\beta-1} \\ 0_\beta &= \lambda_{\beta-1} \end{aligned} \tag{18}$$

in accordance with (15).

A modified definition of dimension is next identified on the basis of analogy with the classical definition of dimension in *Euclidean* space. We consider the *Cartesian* coordinate in *Euclidean* space $(x, y, z) = (N_x, N_y, N_z)$ and note that the with discrete spectrum shown in Fig.3

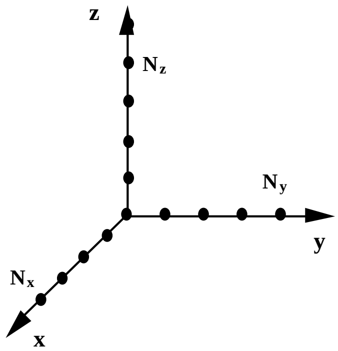

Fig.3 Discrete Cartesian coordinates in three-dimensional Euclidean space.

the total number of atoms in the volume is $V = N = xyz = N_x N_y N_z$. Hence, a three-dimensional *Cartesian* coordinates means that there are three degrees of freedom that can accommodate atoms at equal intervals under a constant measure in a *Euclidian* geometry. Following this same model, since each element at scale β+1 contains $N_{AE\beta+1}$ atoms that constitute $N_{ES\beta}$ independent elements of the system at lower scale β each of which will in turn decompactify into $N_{AE\beta}$ atoms (Fig.2), the total number of atoms of the system will be given by

$$N_{AS\beta} = \left(N_{AE\beta}\right)^{N_{ES\beta}} \tag{19}$$

In the following it will be shown that $N_{ES\beta} = N_{AE\beta+1}$ could be identified as the fractal dimension at scale β+1. By (13) and (19), the system length at scale β becomes

$$\begin{aligned} L'_{\infty\beta} = \ln N_{AS\beta} &= \ln(N_{AE\beta})^{N_{ES\beta}} \\ &= N_{ES\beta} \ln(N_{AE\beta}) \end{aligned} \tag{20}$$

By definitions in (14) the result (20) can also be expressed as

$$L'_{\infty\beta} = N_{ES\beta}\lambda_\beta \tag{21}$$

The relationship between the element and the system lengths of adjacent scales in (15) suggests that

$$\begin{aligned} \lambda_\beta = \ln N_{AE\beta} = L'_{\infty\beta-1} &= N_{ES\beta-1}\lambda_{\beta-1} \\ &= N_{ES\beta-1} \ln N_{AE\beta-1} = \ln\left(N_{AE\beta-1}\right)^{N_{ES\beta-1}} \\ &= \ln N_{AS\beta-1} \end{aligned} \tag{22}$$

such that

$$N_{AS\beta-1} = \left(N_{AE\beta-1}\right)^{N_{ES\beta-1}} \tag{23}$$

in accordance with (19).

In view of the physical nature of the statistical fields shown in Fig.1, it is reasonable to expect that the mathematics required for their description should involve dimensional entities like other branches of physical science. Therefore the element length λ_β will be defined as the unit or "measure" such that by (13) and (14) one can introduce the "dimesionless" or "measureless" coordinate

$$x_\beta = \frac{x'_\beta}{\lambda_\beta} = \frac{\ln N_{A\beta}}{\ln N_{AE\beta}} \tag{24}$$

that by (19) gives

$$x_\beta = \frac{N_{E\beta} \ln N_{AE\beta}}{\ln N_{AE\beta}} = N_{E\beta} \tag{25}$$

Therefore, the natural numbers 1, 2, 3, … of a *Euclidean* space at scale β refer to integral numbers of this *constant* "measure" λ_β defined in (14). Hence, without specification of a "*measure*" the natural numbers will have no meaning in mathematics in the same spirit that numbers have no meaning without specification of their "*dimension*" in physical sciences. The conventional operations of addition, subtraction, division and multiplication will apply to such measureless numbers. However, such a *Euclidian* space with a constant measure will transform into a generalized *Riemannian* space if the measure λ_β becomes a variable. The distribution of particles amongst various energy levels of such statistical fields and their connection to quantum mechanics was recently described in terms of a scale invariant *Planck* distribution function [35].

With the "measureless" coordinate (24) the cascade (15) will become *normalized*

$$\begin{array}{l}
\quad \cdots \\
L_{\infty\beta} \,\rule{4cm}{0.4pt}\, 1_\beta \,\rule{1.5cm}{0.4pt}\, 0_\beta \\
\qquad\qquad\qquad\qquad\qquad L_{\infty\beta-1} \,\rule{0.8cm}{0.4pt}\, 1_{\beta-1} \,\rule{0.4cm}{0.4pt}\, 0_{\beta-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots
\end{array} \tag{26}$$

It is important to note that in (26), the interval $(1_{\beta-1}, 0_{\beta-1})$ is considered to be unobservable at the scale β and is only revealed after re-scaling as a result of decompactification of 0_β to be further discussed in the following section. Also, rather than the factor of 10 between different decimal generations in the classical arithmetic, the interval $(1_\beta, 0_\beta)$ opens to an infinite interval $(L_{\infty\beta-1}, 0_{\beta-1})$ at the first lower scale of β-1 as shown in (15) and (26).

It is interesting to examine the connection of the result (23) with the classical definition of fractal dimension [36]

$$D = -\frac{\ln N(r)}{\ln(r)} \tag{27}$$

where N(r) is the total number of unit shapes and r is the size of the coarse–graining. From (19)-(23), one can introduce a scale-invariant definition of fractal dimension as [37]

$$D_\beta = N_{AE\beta} = N_{ES\beta-1} = \frac{\ln N_{AS\beta-1}}{\ln(N_{AE\beta-1})} \tag{28}$$

that by (27) leads to the invariant definition of coarse-graining

$$r_\beta = \frac{1}{N_{AE\beta-1}} \tag{29}$$

Since each element could contain very large number of "atoms", the fractal dimensions of typical statistical field shown in Fig.1 could be exceedingly large, 10^7.

4 Implications to Nonstandard Analysis and the Existence of Infinitesimals

The classical problem of the existence of infinitesimals in mathematics is generally addressed within the framework of nonstandard analysis of *Robinson* [38] and internal set theory IST of *Nelson* [39]. The implication of these theories to the resolution of the classical problem of *Zeno's* paradox has been discussed [40]. It is interesting to examine the implications of the model discussed in the previous section to the theory of nonstandard analysis.

The problem of existence of infinitesimals is often discussed in connection to the classical solution of *Nicholas* of *Cusa* connecting the area and circumference of a circle [41]. A circle of unity radius is decomposed into infinitely many straight-line segments of equal lengths. The circle area is the sum of the areas of infinite triangles all of which have altitude 1. For each triangle the area is half the base times the altitude. Therefore, the area of the triangles is half of the sum of the bases. Since the area of the circle is the sum of the areas of triangles and the sum of the bases is the circumference of the circle, the area of a circle of unity radius is equal to half its circumference that is a true conclusion. However, at this point *Berkeley's* classical objection is mentioned [41] that the infinitesimal base is either zero thus leading to zero area, or else is finite and the sum of infinite number of which will lead into infinite area.

Because in the classical solution of *Nicholas* infinite number of triangles is considered the base of the equilateral triangles should be taken as equal to the arc length $rd\theta$ such that

$$A = \pi r^2 = \sum_{\infty} (r/2) r d\theta \tag{30}$$

Now, similar to the linear coordinate, one assumes that the angular coordinate θ around the circle is infinitely divisible such that for r = 1 and $d\theta = 2\pi / N$

$$A = \lim_{N\to\infty} \sum_{N=1}^{N} (1/2)(\frac{2\pi}{N}) = C/2 = \pi \tag{31}$$

in accordance with the *Archimedes'* method of exhaustion [41].

Some implications of the scale-invariant model of coordinates introduced in the previous section on the nonstandard analysis [38, 39] are discussed next. For this purpose, the qualitative model of the cascade of coordinate shown in (26) above must be quantitatively described. The problem involves the identification of a particular distribution function for the "atoms" at the scale $\beta - 1$ such that an infinite number of them $(\infty_{\beta-1}, 1_{\beta-1}, 0_{\beta-1})$ will fit the finite interval $(1_\beta, 0_\beta)$. Because of the fact that all of the statistical fields shown on the left hand column of Fig.1, involve equilibrium and hence random distributions of particles it is natural to expect that their description should involve *Gaussian* normal distribution.

Since the unity of scale β relates to infinity of the adjacent scale $\beta - 1$, we define the scale invariant "measureless" coordinate as

$$x_\beta = \frac{x'_\beta}{\lambda_\beta} = N_{ES\beta} \tag{32}$$

that has been "non-dimesionalized" in terms of the "measure" λ_β defined by

$$\lambda_\beta = L_{+\infty\beta-1} = \int_{0_{\beta-1}}^{\infty_{\beta-1}} e^{-x_{\beta-1}^2} dx_{\beta-1} = \frac{\sqrt{\pi_{\beta-1}}}{2} \tag{33}$$

In view of the relation (32)-(33), the range $(0_\beta, 1_\beta)$ of the outer coordinate x_β will correspond to the range $(0_{\beta-1}, \infty_{\beta-1})$ of the inner coordinate $x_{\beta-1}$ as in (26) leading to the coordinate hierarchy schematically shown in Fig.4.

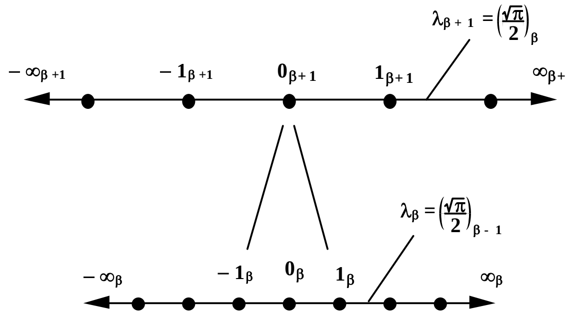

Fig.4 Hierarchy of normalized coordinates associated with cascades of embedded statistical fields.

The occurrence of the important number π in the size of the measure (33) is related to the choice of the normal density $e^{-x_\beta^2}$ made in harmony with the random nature of the distribution of the "atoms". This is because this same measure occurs in the normalization of the *Gauss's* error function

$$Erf(\eta_\beta) = \frac{2}{\sqrt{\pi_\beta}} \int_{0_\beta}^{\eta_\beta} e^{-x_\beta^2} dx_\beta \tag{34}$$

corresponding to the particular scale β. Therefore, the number π_β in (33)-(34) also involves the subscript β to emphasize that the unit of this important number also depends on the specific "measure" being employed to describe the physical system such as the typical units

$$\begin{array}{ll} 3.1415\ldots & \text{millimeters} \\ 3.1415\ldots & \text{meters} \\ 3.1415\ldots & \text{kilometers} \end{array} \tag{35}$$

It is important to note that when one moves from the outer scale $\beta+1$ into the inner scale β, the zero measure $0_{\beta+1}$ will decompactify and generate a new finite interval $(-1_\beta, 1_\beta)$ as schematically shown in Fig.4. For example, if the size of an atom at scale $\beta + 1$ is one millimeter with zero measure $0_{\beta+1} = \ln(1_\beta)$, when one moves to a lower scale with atomic size 10^{-6} m, the atom of $0_{\beta+1}$ becomes 1000 atoms of scale β thus leading to a finite normalized interval $(-1_\beta, 1_\beta)$ as schematically shown in Fig.4. In

Fig.4, it appears that finite completed intervals have been constructed by adding an infinite number of elements thus contradicting *Gauss* [42]

> **"I object to the use of an infinite magnitude as something completed; this is never admissible in mathematics. One must not interpret infinity literally when, strictly speaking, one has in mind a limit approached with arbitrary closeness by ratios as other things increase without bounds."**

However, in view of the transcendental nature of the number π occurring in (33) the intervals are not strictly speaking ever completed.

It is suggested that nonstandard numbers [38, 39] at scale β be identified as all the numbers within the range

$$0_\beta > x_{\beta-1} > 0_{\beta-1} \tag{36}$$

or smaller that are non-observable at the scale β and are only revealed and hence become measurable at the lower scale β–1 because they are produced through decompactification of the zero-measure of scale β. As an example, our galaxy the *Milky Way* that constitutes a point or "atom" at cosmic scale β = g (Fig.1) when approached will decompactify into billions of stars constituting new atoms thus revealing its complex internal coordinate structure. It is the possibility of such internal structures that allows the *Gaussian* curvature to be an *intrinsic property* of surfaces and lines at ever-smaller scales. All numbers in the range (36) at scale β–1 are nonstandard since they will be smaller than all numbers at the scale β no matter how small.

Next, infinitesimal numbers for the scale β could be identified as numbers belonging to the scale β–2 or smaller in the range

$$0_\beta = L_{\infty\beta-2} > x_{\beta-2} > 0_{\beta-2} \tag{37}$$

This is because even an infinite number of elements $\lambda_{\beta-2}$ at the scale β–2 only produce $L_{\infty\beta-2}$ that is a number less than all numbers of scale β and as such could be considered as infinitesimal.

If for the scale invariant definition of "measure" in (33) one uses the more conventional larger density $e^{-x_\beta^2/2}$, then one obtains for the total number of "atoms" on the infinite line $(-\infty_\beta, \infty_\beta)$

$$L_{\pm\infty\beta} = N_{L\pm\infty_\beta} = \int_{-\infty_\beta}^{\infty_\beta} e^{-x_\beta^2/2} dx_\beta = \sqrt{2\pi_\beta} \tag{38}$$

Similarly, the total number of "atoms" on the infinite plane formed by two infinite lines of constant normal density $e^{-x_\beta^2/2}$ will be

$$A_{\infty\beta} = N_{A_{\infty\beta}} = \int_{-\infty_\beta}^{\infty_\beta} e^{-x_\beta^2/2} dx_\beta \int_{-\infty_\beta}^{\infty_\beta} e^{-y_\beta^2/2} dy_\beta$$
$$= \int_{-\infty_\beta}^{\infty_\beta} e^{-(x_\beta^2+y_\beta^2)/2} dx_\beta dy_\beta = \int_{0_\beta}^{\infty_\beta}\int_{0_\beta}^{2\pi_\beta} re^{-r_\beta^2/2} dr d\theta = 2\pi_\beta \tag{39}$$

Finally, the total number of "atoms" in the infinite volume formed by the three infinite lines with constant density $e^{-x_\beta^2/2}$ will be

$$V_{\infty\beta} = N_{V_{\infty\beta}} = \int_{-\infty_\beta}^{\infty_\beta}\int_{-\infty_\beta}^{\infty_\beta}\int_{-\infty_\beta}^{\infty_\beta} e^{-(x_\beta^2+y_\beta^2+-z_\beta^2)/2} dx_\beta dy_\beta dz_\beta$$
$$= 2\int_{0_\beta}^{\infty_\beta}\int_{0_\beta}^{2\pi}\int_{0_\beta}^{\pi/2} r_\beta^2 e^{-r_\beta^2/2} \sin\theta dr_\beta d\phi_\beta d\theta_\beta = (2\pi_\beta)^{3/2} \tag{40}$$

The total number of atoms in infinite line (38), plane (39), and volume (40)

$$L_{\pm\infty\beta} = (2\pi_\beta)^{1/2} \tag{41a}$$

$$A_{\infty\beta} = 2\pi_\beta \tag{41b}$$

$$V_{\infty\beta} = (2\pi_\beta)^{3/2} \tag{41c}$$

in the limit of smallest possible scale $\beta \to 0$ when space is composed of atoms of zero size will correspond to the monads of *Leibniz*. The results (41) for infinite space could be compared with the classical results for finite domain corresponding to unity density

$$C(r) = \int_0^{2\pi} r d\theta = 2\pi r \tag{42a}$$

$$A(r) = \int_0^r\int_0^{2\pi} r dr d\theta = \pi r^2 \tag{42a}$$

$$V(r) = \int_0^r\int_0^{2\pi}\int_0^{\pi/2} r^2 \sin\theta dr d\phi d\theta = \frac{4}{3}\pi r^3 \tag{42a}$$

One notes that the length of the circumference of the circle of infinite radius in (41a) involves the square root $(2\pi_\beta)^{1/2}$ while the classical result (42a) involves 2π. The occurrence of π_β in (41b) and (42b) is to be expected since area is related to the square of length. The reason for difference between (41a) and (42a) is now further examined.

It is suggested that the occurrence of different powers of π_β in (41a) and (42a) could be caused by an unintended change of measure in the classical results. To show this, the modified expression for the circumference of the circle (38) is written with unity density, i.e. for monads of *Leibniz* as atoms, parallel to (42a) as

$$C(r') = \int_0^{\sqrt{2\pi}} r'd\theta' + \int_{-\sqrt{2\pi}}^{0} r'd\theta'$$
$$= 2\int_0^{\sqrt{2\pi}} r'd\theta' \qquad (43)$$

The circumference in (43) corresponds to two semi-circles of unity radius at scale β+1 that is constructed by attaching the two ends of the doubly infinite interval $(-\infty_\beta, \infty_\beta) = (-1_{\beta+1}, 1_{\beta+1})$ as schematically shown in Fig.5.

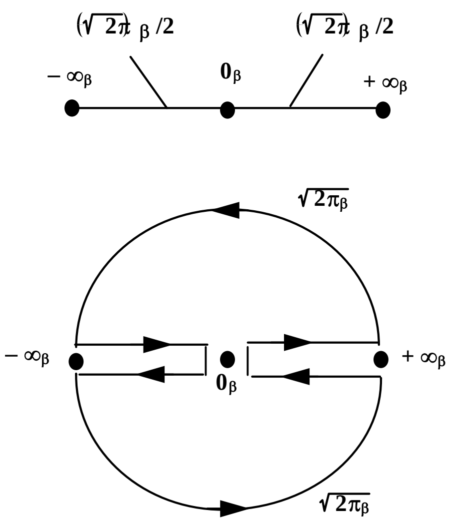

Fig.5 Transformation of doubly infinite line $(-\infty_\beta, \infty_\beta)$ into a circle at scale β.

As shown in Fig.5, the angle θ' changes by $\sqrt{2\pi_\beta}$ as one moves around the semi circle. However, one now introduces the modified new measures for length and angle

$$r_\beta = r'_\beta / (\sqrt{2\pi_\beta}/2) = r'_\beta / \lambda_\beta \quad , \quad \theta_\beta = \sqrt{2\pi_\beta}\ \theta'_\beta \qquad (44)$$

where λ_β is obtained from the new density $e^{-x_\beta^2/2}$ as

$$\lambda_\beta = \int_{0_\beta}^{\infty_\beta} e^{-x_\beta^2/2} dx_\beta = \sqrt{\pi_\beta / 2} \qquad (45)$$

After substitutions from (44) the result (43) assumes the classical form

$$C(r_\beta) = \int_0^{\sqrt{2\pi_\beta}} r'/(\sqrt{2\pi_\beta}/2)\sqrt{2\pi_\beta}\, d\theta'_\beta$$
$$= \int_0^{2\pi_\beta} r_\beta d\theta_\beta = 2\pi_\beta r_\beta \qquad (46)$$

With the modified definitions in (44) the absence of the square root difference between lengths (41a) versus area (41b) in the classical result (42a)-(42b) is reconciled. Similar to (46), the definitions (44) lead to the classical expressions for the area (42b) and the volume (42c).

With identical density, the area of the circle covering the infinite plane given by (39) will be exactly equal to the area of the infinite square with the side $L_{\pm\infty\beta} = \sqrt{2\pi_\beta}$ given by (38) covering the same infinite plane as schematically shown in Fig.6.

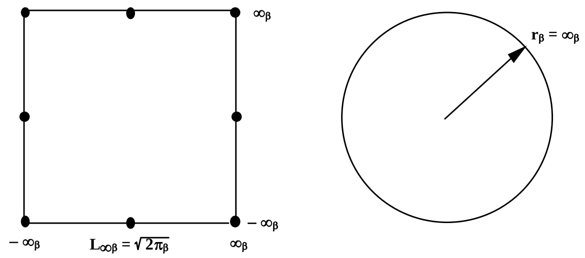

Fig.6 Squaring the circle by transforming an infinite square $(-\infty_\beta, \infty_\beta)$ into an infinite circle.

It is reasonable to expect that one could also square a circle in a *finite domain* if one allows the density to be a variable and hence work in a variable-measure generalized *Riemannian* space rather than the *Euclidian* one as in Fig.6.

5 Invariant Form of the Wave Equation

An important aspect of the analysis discussed in the previous section concerns its possible connection to the *Fourier* representation of generalized functions. Therefore, it is interesting to apply the scale invariant forms of the conservation equations (10)-(12) to derive an invariant form of the wave equation. For any equilibrium statistical field at scale β shown on the left hand column of Fig.1, we consider small velocity perturbations, “sound waves”, at the lower scale β–1 that are described by the equation of motion at the yet smaller scale β–2. For example, for conventional gas dynamics that corresponds to the scale of equilibrium cluster dynamics ECD β = c, sound waves are perturbations of velocity of molecules (β–1 = m) that follow the equation of motion (12) at the atomic scale (β – 2 = a) that in the absence of reactions $\Omega_\beta = 0$ becomes

$$\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{w}_a.\nabla \mathbf{v}_a = \nu_a \nabla^2 \mathbf{v}_a - \frac{\nabla p_a}{\rho_a} \qquad (47)$$

At the large scale of ECD, the pressure gradient term and the viscous term $\nu_a \simeq 0$ that are at the exceedingly local LAD scale will be neglected and the latter assumption is equivalent to neglecting all dissipative effects. If the characteristic length λ'_m and velocity v'_m of LMD scale are applied to non-dimensionalize (47) one would have the inverse of the *Reynold's* number at molecular scale $Re_m = \nu_a / L'_a w'_a = \nu_a / \lambda'_m v'_m$ multiplying the viscous term, the dimensionless pressure will become $p_a = p'_a / \rho_a v'^2_m$, and the dimensionless time $t = t' v'_m / \lambda'_m$. Then, the forgoing assumptions will relate to the limits $Re_m \gg 1$ and $\nabla p_a \ll 1$.

With the assumptions mentioned above (47) reduces to

$$\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{w}_a.\nabla \mathbf{v}_a = 0 \qquad (48)$$

that in view of (1)-(2) becomes

$$\frac{\partial u_m}{\partial t} + v_m.\nabla u_m = 0 \qquad (49)$$

The mean molecular velocity is identified with the speed of sound $v_m = a$ that will be a constant for a given temperature of the gas. Taking the first time derivative of (49) gives

$$\frac{\partial^2 u_m}{\partial t^2} + v_m.\nabla \frac{\partial u_m}{\partial t} = 0 \qquad (50)$$

Substituting in (50) for $\partial u_m / \partial t$ from (49) one obtains the wave equation for the propagation of acoustic waves in atmosphere

$$\frac{\partial^2 u_m}{\partial t^2} = v_m^2 \nabla^2 u_m \qquad (51)$$

The above procedure when applied to the mass and energy conservation equations (10) and (11) in the absence of dissipations lead to the wave equation for propagation of density and thermal signals [27]

$$\frac{\partial^2 \rho_a}{\partial t^2} = v_m^2 \nabla^2 \rho_a \qquad (52)$$

$$\frac{\partial^2 T_a}{\partial t^2} = v_m^2 \nabla^2 T_a \qquad (53)$$

Finally, by application of procedures identical to the steps (47)-(51) one obtains for any equilibrium statistical field at scale β the invariant wave equation

$$\frac{\partial^2 u_\beta}{\partial t^2} = v_\beta^2 \nabla^2 u_\beta \qquad (54)$$

Since each statistical field within the hierarchy (Figs.1, 2) has an "atomic" $\mathbf{u}_\beta$ and an element $\mathbf{v}_\beta$ velocity, in view of (54) one may associate a "wave" and a "particle" speed with each statistical field

$\mathbf{v}_\beta = \mathbf{u}_{\beta+1}$	**Wave speed**	(55a)
$\mathbf{u}_\beta = \mathbf{v}_{\beta-1}$	**Particle speed**	(55b)

For the statistical fields ECD, β = c by (51)

$\mathbf{v}_m = \mathbf{u}_c = \mathbf{a}$	**Acoustic waves**	(56a)
$\mathbf{u}_m = \mathbf{v}_a$	**Acoustic radiation**	(56b)

where the speed of sound in standard atmosphere is $\mathbf{v}_m$ = a = 350 m/s. The velocity of acoustic radiation or the mean atomic speed will be equal to the speed of typical detonation waves [43] that is about $\mathbf{u}_m$ = $\mathbf{v}_a$ = 2000 m/s. When (54) is applied to the scale LKD (Fig.1), one arrives at the wave equation for motion of tachyons that propagate at superluminal speeds $\mathbf{u}_t \geq 2\times10^{10}c$ [44-46].

6 Concluding Remarks

A scale-invariant model of statistical mechanics was applied to examine the physical foundation of nonstandard analysis and to identify the nature of nonstandard numbers and to establish the existence of infinitesimals. An invariant logarithmic definition of coordinate was presented and the concept of "measureless" or "dimensionless" numbers was described. Also, a scale-invariant definition of fractal dimension was introduced. Finally, invariant forms of conservation equations were used to derive an invariant wave equation that is applicable to both electromagnetic as well as gravitational waves.

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