Numerical Scheme of Magnetic Monopoles

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Abstract: In this paper, we present a numerical method to compute the 't Hooft-Polyakov static magnetic monopoles as an asymptotic limit of a coupled system of evolution equations. An efficient numerical scheme and its results will be presented.

Key–Words: Monopoles, Yang-Mills-Higgs equations, Numerical Scheme

1 Introduction

We consider magnetic monopoles in non-Abelian gauge theories which were discovered by 't Hooft [18] and Polyakov [14] as solutions of coupled Yang-Mills-Higgs systems. The dynamical magnetic monopoles are described by the equations of motion derived from the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi_a)^2 + \beta(\frac{1}{2}\phi_a^2 - \frac{1}{8}(\phi_a^2)^2) \quad (1)$$

Here ϕ_a is a scalar field ($a = 1, 2, 3$), $F^a_{\mu\nu}$ is the tensor of the Yang-Mills field A^a, D_μ is the covariant derivative:

$$\begin{aligned} F_{\mu\nu} &= \bigtriangledown_\mu A_\nu - \bigtriangledown_\nu A_\mu + ig[A_\mu, A_\nu] \\ D_\mu\phi &= \bigtriangledown_\nu\phi + ig[A_\mu, \phi] \end{aligned} \quad (2)$$

where [,] denotes the product in $su(2)$; g and β are coupling constants. The equations of motion in dimensionless quantities are

$$\begin{aligned} D_\mu D^\mu\phi &= -\lambda\phi(\mid\phi\mid^2 - 1) \\ D_\nu F^{\mu\nu} &= [\phi, D^\mu\phi] \end{aligned} \quad (3)$$

In this paper, we restrict our study to spherically symmetric configuration solutions of the finite energy functional

$$\begin{aligned} L(\lambda) = \int_0^\infty [(S_r)^2 &+ \frac{1}{2}\left(\frac{S}{r}\right)^2 (S-2)^2 \\ &+ \frac{1}{2}r^2R_r^2 + R^2(1-S)^2 \\ &+ \frac{\lambda}{8}r^2(1-R^2)^2]dr \end{aligned} \quad (4)$$

The functions S and R are found as the minimum of the functional (4) as demonstrated in [4] . The Euler-Lagrange variational derivation of (4) leads to the following coupled ordinary differential equations

$$\begin{aligned} S_{rr} &- \frac{1}{r^2}S(1-S)(2-S) + (1-S)R^2 = 0 \quad (5) \\ R_{rr} &+ \frac{2}{r}R_r - \frac{2}{r^2}R(1-S)^2 + \frac{\lambda}{2}R(1-R^2) = 0 \end{aligned}$$

The functions $S(r)$ and $R(r)$ must satisfy the boundary conditions at the origin and at infinity:

$$\begin{aligned} &R(0) = 0 \qquad S(0) = 0 \\ &R(r) \to 1 \qquad S(r) \to 1 \quad \text{as } r \to \infty \end{aligned} \quad (6)$$

The conditions (6) are consequence of the symmetry and the continuity of $S(r)$ and $R(r)$ at the center of the monopole. The asymptotic behavior of these functions can be described by expanding their Taylor's series around $r = 0$ and at $r \to \infty$. The asymptotic limits of the functions $S(r)$ and $R(r)$ with critical value λ are given by:

$$\begin{aligned} S(r) &= O(r^2) && \text{for } r \to 0 \\ R(r) &= O(r) && \text{for } r \to 0 \\ S(r) &= 1 + O(e^{-r}) && \text{for } r \to \infty \\ R(r) &= 1 + O(e^{-\lambda r}) && \text{for } r \to \infty \end{aligned}$$

In this paper we introduce an efficient numerical method for solving the system of differential equations (5). Our approach is based first on the transformation $S(r) = r\sigma(r)$ that reduces the order of the solution at the center of the monopole. This allows us to construct a finite difference scheme to compute the solution, with any desired degree of accuracy, while conserving the energy. Second this approach is based on implementing the gradient flow for minimizing the energy functional using finite difference scheme.

As a historical remark, we point out that the first calculation of the magnetic monopoles were performed by Bogomolnyi and Marionov [6]. Their methods is based on integrating the system (5) using a quasilinearization approach (Newton's method) and iterating until the difference between computed energies are acceptable. Ajithkumar and Sabir [2] described numerical approximation to obtain the solutions of the monopoles in the form of power series involving a large number of arbitrary constants with some limitations.

2 Numerical Scheme

We had introduced the variational discretization of a similar Lagrangian to (4) in connection with the computations of magnetic vortices. The solutions of the derived equations are the minimum of the corresponding energy [1]. In this section, we introduce the reduction transformation and the gradient flow evolution equations, then we present the finite difference scheme with a proof of its linearized stability.

2.1 Reduction Transformation

The function $S(r)$ approaches zero in the order of r^2, so we reduce the order of $S(r)$ by one using the transformation

$$S(r) = r\sigma(r) \tag{7}$$

with the corresponding boundary conditions

$$\begin{aligned} \sigma(0) &= \sigma(\infty) = 0 \qquad (8) \\ \int_0^\infty \partial_r(\sigma^2)dr &= \sigma^2|_0^\infty = 0 \end{aligned}$$

Substitution (7) is essential for the stability of the proposed discretization scheme to solve the system of equations (5). Using the substitution (7) and the identity

$$(\partial_r S)^2 = \sigma^2 + 2r\sigma\sigma_r + r^2\sigma_r^2 \tag{9}$$

in the Lagrangian (4) and the finite energy assumption. Next step, we use the gradient flow approach

$$(R_t, \sigma_t) = \delta L_{(R,\sigma)}(R, R_r, \sigma, \sigma_r)$$

to derive the variational equations

$$\begin{aligned} \sigma_t = r\sigma_{rr} &+ 2\sigma_r - \frac{1}{r}\sigma(1-r\sigma)(2-r\sigma) \\ &+ (1-r\sigma)R^2 \end{aligned}$$

$$\begin{aligned} R_t = rR_{rr} &+ 2R_r - \frac{2}{r}R(1-r\sigma)^2 \\ &+ \frac{\lambda}{2}rR(1-R^2) \end{aligned} \tag{10}$$

2.2 Finite Difference Scheme

We point out that the energy functional $L(R, R_r, \sigma, \sigma_r)$ is positive definite and it has a unique minimum [1]. This minimum is the stationary limit of the nonlinear parabolic system of equations

$$\begin{aligned} \frac{d}{dt}\sigma &= -\delta L_\sigma \\ \frac{d}{dt}R &= -\delta L_R \end{aligned} \tag{11}$$

We introduce a second order implicit scheme, for which the functions $R(r)$ and $\sigma(r)$ are the evolution limits:

$$\begin{aligned} \frac{3\sigma_m^{n+1} - 2\sigma_m^n}{2\Delta t} - \frac{2\sigma_m^n - \sigma_m^{n-1}}{2\Delta t} &= \\ r_m\frac{\sigma_{m+1}^{n+1} - 2\sigma_m^{n+1} + \sigma_{m-1}^{n+1}}{h^2} &+ G_\sigma(\sigma^n, R^n) \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{3R_m^{n+1} - 2R_m^n}{2\Delta t} - \frac{2R_m^n - R_m^{n-1}}{2\Delta t} &= \\ r_m\frac{R_{m+1}^{n+1} - 2R_m^{n+1} + R_{m-1}^{n+1}}{h^2} &+ G_R(\sigma^n, R^n) \end{aligned} \tag{13}$$

where $\sigma_m^n = \sigma(n\Delta t, mh)$, $R_m^n = R(n\Delta t, mh)$, and G_σ and G_R are nonlinear remaining terms from L_σ and L_R evaluated at time step $n\Delta t$ as follows

$$\begin{aligned} G_\sigma &= 2\frac{\sigma_{m+1}^n - \sigma_{m-1}^n}{2\Delta r} \\ &- \frac{1}{r_m}\sigma_m^n(1 - r_m\sigma_m^n)(2 - r_m\sigma_m^n) \\ &+ (1 - r_m\sigma_m^n)(R_m^n)^2 \\ G_R &= 2\frac{R_{m+1}^n - R_{m-1}^n}{2\Delta r} \\ &- \frac{2}{r_m}R_m^n(1 - r_m\sigma_m^n)^2 \\ &+ \frac{\lambda}{2}r_m R_m^n(1 - (R_m^n)^2) \end{aligned} \tag{14}$$

Proposition 1 *The implicit scheme (12-13) is unconditionally stable and it is of order $O(\Delta t^2) + O(h^2)$.*

Proof: We linearize the equations (12 and 13) and apply the Fourier transform by replacing σ_m^n and R_m^n by $\xi^n e^{im\beta h}$. We obtain the quadratic equation

$$(3 + 4\alpha r_m sin^2(\beta h/2))\xi^2 - 4\xi + 1 = 0$$

for both equations (12 and 13) . Solving for ξ we get

$$\xi = \frac{2 \mp \sqrt{1 - 4\alpha r_m sin^2(\beta h/2)}}{2 + 1 + 4\alpha r_m sin^2(\beta h/2)} = \frac{2 \mp \sqrt{1-w}}{2+1+w}$$

If $1 - w$ is nonnegative, then we have

$$|\xi| \leq \frac{2 \mp \sqrt{1-w}}{2+1+w} \leq 1$$

if $1 - w$ is negative, then

$$\xi = \frac{2 \mp i\sqrt{w-1}}{2+w+1}$$

for which

$$\begin{aligned} |\xi|^2 &= (\frac{2}{2+w+1})^2 + \frac{w-1}{(2+w+1)^2} \\ &= \frac{4+w-1}{(2+w+1)^2} \leq 1 \end{aligned}$$

Thus for any value of α or β both roots are bounded by 1. This proves the unconditional stability of (12 and 13). The order of the scheme is derived from the construction of the finite difference algorithm. $\square$

The discretized approximations of the system of equations based on this scheme lead to tridiagonal system of equations with variable entries depending on the distance r_m from the center of the monopole and has the form

$$\mathbf{X} = \begin{pmatrix} \gamma_1 & \alpha_1 & 0 & 0 & \ldots & 0 & 0 \\ \alpha_1 & \gamma_2 & \alpha_2 & 0 & \ldots & 0 & 0 \\ 0 & \alpha_2 & \gamma_3 & \alpha_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \alpha_{n-1} & \gamma_n \end{pmatrix}$$

where $\alpha_i = -\frac{2\Delta t}{\Delta^2 r} \cdot 2r_i$ and $\gamma_i = 3 + 2\alpha_i$.
These equations are solved by standard techniques. We terminate the iteration process whenever the difference between consecutive energy values becomes smaller than a preassigned number. For example, when $h = 0.01$, we terminate the iterations when the difference is less than 10^{-8}, that is

$$|L(\sigma^{n+1}, R^{n+1}) - L(\sigma^n, R^n)| < 10^{-8}$$

In Figure 1 we present the numerical solution $R(r)$ of the magnetic monopoles corresponding to the parameters $\lambda = 0.1, 1.0, 100$ where the graph with largest slope corresponds to $\lambda = 100$. In Figure 2 we present the numerical solution $S(r)$ of the magnetic monopoles corresponding to the the parameters $\lambda = 0.1, 1.0, 100$ where the graph with largest slope corresponds to $\lambda = 100$.

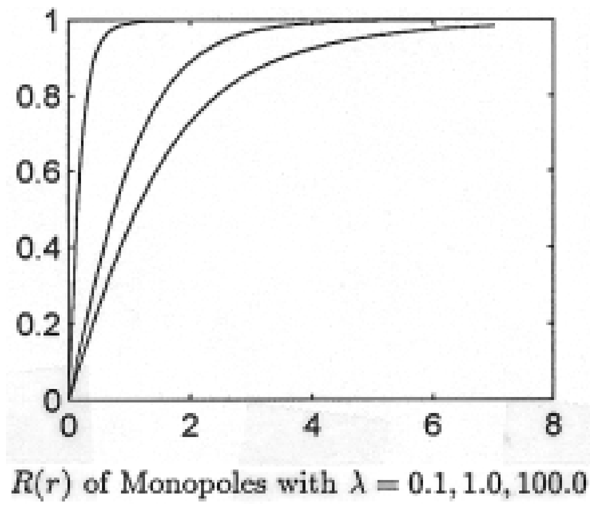

Figure 1: *R(r) for* $\lambda = 0.1, 1, 100$

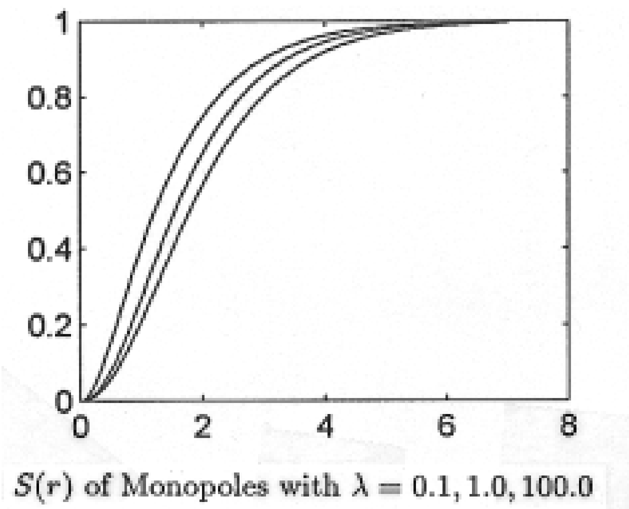

Figure 2: *S(r) for* $\lambda = 0.1, 1, 100$

3 CONCLUSION

In this paper, we have presented a novel numerical method to solve the second-order field equations of the 't Hooft-Polyakov magnetic monopole theory. This numerical scheme was based on the motion of gradient flows approach in the form of an implicit finite difference scheme. This developed approach can be applied to compute various monopole configurations of Yang-Mills equations and vortices of the Ginzburg-Landau equations of superconductivity. In addition, a stability proof of the numerical scheme was given. Our numerical results were demonstrated graphically for magnetic monopoles of any multiplicity. The computed solutions in this paper can be used as initial data to study the dynamic evolution of monopoles and vortices.

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