

On Arrow Paradox: Preference Ranking and Decision Making Mechanism Based on Rating Method and Expert Information.

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Abstract: - This paper deals with group decisions based on the rating methods of fuzzy and regular preference rankings. The preference ranking is one of the methods to solve so-called selection problems. Selection problems are very important for decision making in unique systems such as medical, environmental or ecological ones. Very often the right decision is based upon expert information. In this paper we deal with some approaches to the choice of the best variants [1], and its application to the multi-criteria optimization and decisions. This paper presents axiomatic systems of rating methods of preference ranking based on fuzzy expert information. Results include the convergence of consensus ranking to the real ranking almost everywhere, the inclusion of the consensus ranking into the Kemeny Median set [2], and that consensus ranking almost always (with probability one) satisfies all five Arrow axioms. It is shown, that contemporary rating systems, (for instance those used in sports classifications, [3]), are congruent in the sense of producing the same final preference ranking.

Key-Words: - Arrow Paradox, group decisions, preference ranking, consensus, expert information

1. Preference Ranking.

Let us consider a finite set of objects $A = \{a_1, \dots, a_n\}$ and a set of m experts E . Each expert presents binary matrix $Q_k = (q_{ij})$, $k=1, \dots, m$, and $ij=1, \dots, n$. The problem is to find consensus ranking of objects based upon information provided by experts.

There are a few ways to solve the problem under consideration. The first one is to determine Pareto set $P = \{P / \bigcap Q_k \subseteq P \subseteq \bigcup Q_k\}$, but set P is too wide. The second method of the solution presented by Arrow [4] was based on a contradictory system of five axioms. The most useful result obtained by Kemeny, is the so-called Kemeny Median: $H = \{K / \sum d(K, Q_k) = \min \sum d(P, Q_k)\}$, which can be determined by methods of integer programming. It is necessary to outline, that the Kemeny Median satisfies four of five Arrow's axioms. Also, there is an inclusion $H \subseteq P$, but still the set H is quite wide.

The methods presented in this paper have three advantages. First, there is a very simple numerical procedure. The second improvement is the possibility of using different forms of expert information such as preference ranking, binary comparison, and multi-comparison at the same time; this significantly differs from previous methods based on uniform types of expert information. The third advantage is that the result of this rating procedure is a unique preference ranking and not just some set of suitable alternatives such as a Pareto Set or a Kemeny Median.

2. Axioms of Rating Systems

Suppose there is some (maybe unknown to the decision-maker) order of objects a_1, \dots, a_n under consideration. Let us assume, that we have chosen some arbitrary scale, and each object has its own yet

unknown value $r_0(a_i)$ in this chosen scale. The following is an axiom of existence:
Axiom 1. There is some order of given objects in any chosen scale.

Let us denote by $\Delta_{ij} = r_0(a_i) - r_0(a_j)$ the difference between real rating values. We believe that binary relationships given by experts satisfy

Axiom 2:

$$\frac{[\text{number of preferences } a_i \succ a_j]}{[\text{number of preferences } a_j \succ a_i]} = f(\Delta_{ij}),$$

where $f(\Delta)$ is a nonnegative and strictly increasing function such that $f(0)=1$.

This assumption shows the odds or fuzzy odds of preferences by experts, who are asked to rank or compare given objects. Let r_{i0} $i = 1, \dots, n$ be some arbitrary initial ratings, r_{ik} is rating of an i -th object after k -th recalculation, and

$$\Delta_{ij}(k) = r_i(k) - r_j(k).$$

The following statement gives the simple way to calculate ratings of objects according to the results of expert estimations.

Axiom 3.

$$\begin{aligned} r_i(k) &= r_i(k-1) + q_{ij}(k) F(\Delta_{ij}(k-1)), \\ r_j(k) &= r_j(k-1) + q_{ji}(k) F(\Delta_{ji}(k-1)), \\ q_{ij}(k) &= \begin{cases} 1, & \text{if } a_i \succ a_j \\ 0, & \text{if } a_i = a_j \\ -1, & \text{if } a_j \succ a_i \end{cases} \end{aligned}$$

In the case of fuzzy information coefficients q are equal values of membership function of a fuzzy binary relationships provided by experts respectively. For fuzzy relationships formulas in axiom 3 should be change as follows:

$$\begin{aligned} r_i(k) &= r_i(k-1) + q_{ij}(k) F(\Delta_{ij}(k-1)) \\ &\quad - q_{ji}(k) F(\Delta_{ji}(k-1)), \\ r_j(k) &= r_j(k-1) + q_{ji}(k) F(\Delta_{ji}(k-1)) \\ &\quad - q_{ij}(k) F(\Delta_{ij}(k-1)) \end{aligned}$$

which means an increase proportional to fuzzy preference and a decrease proportional to fuzzy non-preference.

It is reasonable that for a large Δ_{ij} increasing of r_{ik} should be small, if $a_i \succ a_j$, but decreasing should be large if $a_i \prec a_j$. This idea is very useful, for instance, in some sports methods of classifications: it means that if a strong team or player outs the weak one then there is almost no increasing in the rating for the winner. However in the case of losing the game the higher rated team should loose many points. That is why we have two following assumptions for function F :

Axiom 4. $\lim_{\Delta \rightarrow \infty} F(\Delta) = 0$

Axiom 5. $\lim_{\Delta \rightarrow -\infty} F(\Delta) = L > 0$

Example: The best known rating system has been suggested by Elo [3], and it is used for the ranking of chess players. Elo supposed that 200 points are difference between neighbor grades of players, and that probability of winning of the more qualified player is equal to 0.75. So he had chosen some scale and calculated ratings for all more or less well known players since the beginning of the century. Using the simplest function satisfied axiom 2, which is an exponent, we can determine base of this function:

$$\frac{a^{200}}{a^{200} + 1} = 0.75 \quad \text{and } a = 1.0055$$

The new rating in Elo system is equal to old rating plus ten folded difference between an actual result of the match and expected outcome of the game.

3. Some Results.

The following theorem establishes equivalency for all rating systems of preference ranking including rankings based on fuzzy information.

Theorem 1. For any initial ratings any method based on axioms 1 through 5 presents some preference ranking which is the same as a real unknown ranking with probability one in the space of realization, when $k \rightarrow \infty$.

A sketch of the proof is the following. Let us consider ranking of the two objects a_i and a_j as a rating procedure. Hence we deal with the space of sequences $\Omega = (\omega_1, \omega_2, \dots)$, where $\omega_k = 1$, if $a_i \succ a_j$ and $\omega_k = -1$ otherwise. Each ω is realization of Bernully trials and there is some measure μ in the space Ω (2) which is generated by probability

$$p = \frac{f(\Delta_{ij})}{1 + f(\Delta_{ij})}$$

Suppose that $\Delta_{ij} > 0$, i.e. $p > 1/2$. On the one hand according the weak law of large numbers almost everywhere

$$\lim_{k \rightarrow \infty} [E(\Delta_{ij}(k)) - E(\Delta_{ij}(k-1))] = 0$$

On the other hand

$\Delta_{ij}(k) = \Delta_{ij}(k-1) + 2q_{ij} F(\Delta_{ij}(k-1))$. So with probability one in the space of realization

$$E[\Delta_{ij}] = E[\Delta_{ij} + 2q_{ij} F(\Delta_{ij})],$$

where q_{ij} is a random variable. In such a way we can obtain the following equation with respect to Δ_{ij} :

$$p F(\Delta_{ij}) - (1-p) F(-\Delta_{ij}) = 0.$$

Because of continuity and decreasing of function F and because of $p > 1/2$ there is some positive Δ_{ij} which is solution of last equation. So, if $\Delta_{ij} > 0$, then as a result of rating procedure $\Delta_{ij}(k) > 0$, if $k \rightarrow \infty$ with probability one.

Now we can consider comparison of many objects. Here we deal with matrix

$B_k = (\Delta_{ij}(k))$ $i, j = 1, \dots, n$ and with more complicated ergodic process [6]. The measure in this case is constructed as the measure of Cartesian product of one

dimensional sequences. And if for some a_i and a_j $\Delta_{ij} > 0$ then $a_i \succ a_j$.

Remarks. This theorem shows that all rating systems are equal in the sense that with probability one they would produce the same consensus ranking. Indeed, we have just discussed so called additive rating systems of preference ranking, but the same statements and ideas are correct for multiplicative rating methods in which $r_i(k) = r_i(k-1) * q_{ij}(k) * F(\Delta_{ij}(k-1))$.

4. Ratings and Kemeny Median.

Presume that experts present information about preferences in matrix form: $Q_k = (q_{ij}(k))$, $i, j = 1, \dots, n$, $k = 1, \dots, m$. For any two matrices of binary relations Q_k and Q_l distance between them may be defined in the following way:

$$d(Q_k, Q_l) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n |q_{ij}(k) - q_{ij}(l)|$$

If matrices Q_k present preference ranking by experts then Kemeny median (4) is such matrix K that

$$\sum_{i=1}^m d(K, Q_k) = \min \sum_{i=1}^m d(P, Q_k)$$

The Kemeny median is really a set of such matrices K , and at a present moment it is considered the most useful consensus ranking, however the determination of this median is the problem of integer programming with all difficulties of calculations. The following result establishes connection between Kemeny median and rating rankings:

Theorem 2. The consensus preference ranking obtained as a result of a rating procedure belongs with probability one to the Kemeny Median set in the space of all realizations Ω .

It is easily seen that rating methods could be used as iterative procedures for determining Kemeny Median. The proof of this theorem is based on facts that Kemeny

Median as well as a rating preference ranking are satisfied four of five Arrow axioms.

5. Arrow Paradox

The need to aggregate preferences occurs in many different disciplines: in welfare economics, where one attempts to find an economic outcome which would be acceptable and stable; in decision making, where a person has to make a rational choice based on several criteria; and most naturally in voting systems, which are mechanisms for extracting a decision from a multitude of voters' preferences.

The framework for Arrow's theorem assumes that we need to extract a preference order on a given set of options (outcomes). Each individual in the society (or equivalently, each decision criterion) gives a particular order of preferences on the set of outcomes. We are searching for a preferential voting system, called a *social welfare function*, which transforms the set of preferences into a single global societal preference order. The theorem considers the following properties, assumed to be reasonable requirements of a fair voting method:

Axiom of non-dictatorship: the social welfare function should not simply follow the preference order of a special individual while ignoring all others. This means that the social welfare function is sensitive to more than the wishes of a single voter.

Axiom of unrestricted domain or universality: the social welfare function should create a deterministic, complete societal preference order from every possible set of individual preference orders. In other words: the vote must have a result that ranks all possible choices relative to one another, the

voting mechanism must be able to process all possible sets of voter preferences, and it should consistently give the same result for the same profile of votes—no randomness is allowed in the process.

Axiom of independence of irrelevant alternatives: if we restrict attention to a subset of options and apply the social welfare function only to those, then the result should be compatible with the outcome for the whole set of options. Changes in individuals' rankings of irrelevant alternatives (ones outside the subset) should have no impact on the societal ranking of the relevant subset. This is a restriction on the sensitivity of the social welfare function.

Axiom of positive association of social and individual values: if an individual modifies his or her preference order by promoting a certain option, then the societal preference order should respond only by promoting that same option or not changing, never by placing it lower than before. An individual should not be able to hurt an option by ranking it higher.

Axiom of non-imposition or citizen sovereignty: every possible societal preference order should be achievable by some set of individual preference orders. This means that the social welfare function is onto: It has an unrestricted target space.

Arrow's theorem says that if the decision-making body has at least two members and at least three options to decide among, then it is impossible to design a social welfare function that satisfies all these conditions at once. So far we have the following result.

Conjecture:

The consensus ranking obtained by rating methods in accordance with axioms 1-5 almost always satisfies Arrow axioms in the space of social welfare functions.

Remark: Social welfare function is a function

$F : L(A)^N \rightarrow L(A)$ which aggregates voters' preferences into a single preference order on A. The N-tuple (R_1, \dots, R_N) of voter's preferences is called a *preference profile*.

6. Applications to Multi-objective Optimization.

The problem consists of finding such x from the set of feasible decisions X that gives, in some sense, the best value to objective functions $v_1(x), \dots, v_n(x)$. If the set X is a finite set, then we have a selection problem of decision-making. If X belongs to some continuous space, then we deal with a problem of multi-objective programming.

There are many approaches to the solution of such a problems under consideration. The most effective approaches to the solution are based on determining some utility function $G(v_1(x), \dots, v_n(x))$ of given criteria. The next step is to solve problem of the mathematical programming:

$$\max \rightarrow G(v_1(x), \dots, v_n(x)), \text{ where } x \in X$$

Usually function $G(\cdot)$ depends on preferences of decision-makers or it is based upon expert information. Let us consider some linear-weighted aggregate objective function as an integrated objective function (utility function):

$$G = \alpha_1 v_1(x) + \dots + \alpha_n v_n(x).$$

The question now is how to determine the weight coefficients. Using rating methods of preference ranking, we can ask decision-makers (expert, advisers etc.) to present their preferences of objective functions $v_1(x), \dots, v_n(x)$ in the form of ranking, or binary, or

multi-comparisons, and find coefficients α_k based on final consensus ranking [1, 7].

As a result of just described procedure of preference ranking we can obtain ratings r_1, \dots, r_n for given objective functions. Suppose that we have got $r_1 \geq r_2 \geq \dots \geq r_n$. Let the value of coefficient α_1 to be equal to one, in this case using the structure of the rating procedure we can find other weight coefficients from the following relationship:

$$\frac{\alpha_j}{\alpha_i} = f(\Delta_{ij}), j = 1, \dots, n,$$

where f and Δ_{ij} are defined by just described system of axioms of preference ranking. Respectively, the determined in such a way weight coefficients would give us an opportunity to use additive integrated utility function when we have to deal with few objectives at the same time.

$$G = \alpha_1 v_1(x) + \dots + \alpha_n v_n(x).$$

Now we are able to transform an optimization problem with many objectives to the problem of mathematical programming with only one objective function, and apply one of the developed algorithms to determine optimal solution.

In the conclusion we would like to stress out that rating systems of preference ranking are very flexible. They provide an opportunity to work with different types of expert information such as binary and multi-comparison, ranking, etc. Moreover, there is a possibility to work with fuzzy information. If, for instance, μ_{ij} is measure of belonging that $a_i \succ a_j$, then it is enough to replace q_{ij} by μ_{ij} in axiom 3 to use fuzzy relationship offered by experts. The last remark concerns theorem 1, which holds also under conditions of fuzziness.

Developed in this paper approach was successfully applied in practice. Namely, we tried to address the different priorities of coastal users different criteria were developed mostly by the National Oceanographic and Atmospheric Administration (NOAA) [9], [10] to evaluate the quality of the predictions. The

criteria listed reflect different concerns. The average error will address the possible bias of a model, the absolute error will give information on the overall accuracy of the model, and the standard deviation will give information on the variability of the forecasts. Other, more specialized, criteria such as the positive and negative outlier frequencies will be useful to characterize model performance for unusually high or low water level situations. It is clear that some forecasting methodologies will be better suited for some criteria and worse for others. For example, predictions based on Harmonic Analysis are very good when evaluated by the standard deviation criteria and not as good when using the absolute error criteria. Obtaining such an aggregated criteria for multi-criteria decisions proved itself very useful for the evaluation of the quality of predictions of water levels, as well as for checking the quality of methods to fill gaps in the data collected by the Texas Coastal Ocean Observation Network (TCOON).

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