Embedding the *i*th Johnson Networks into the Hamming Network

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Abstract: - Embedding of graphs is an important and interesting approach to parallel computing. Generally it can be used to model simulation of networks and algorithm structures on different networks. This paper shows that there is an embedding of the Johnson Networks into the Hamming Network.
The vertex set of the Johnson Scheme $G(n,k)$ is the set of all k-subsets of a fixed n-set. Two vertices A and B in $G(n,k)$ are adjacent if $|A\cap B|=k$-1. The *ith* graph of the Johnson Scheme $G_i(n,k)$ is an extension of $G(n,k)$ such that vertices are adjacent if they are they are i–related, that is, if $|A\cap B|=k-i$. i is referred to as the Johnson Distance. The combined *ith* graphs of the Johnson Scheme $SG_i(n)$ is the graph formed by the union of all the graphs $G_i(n,k)$ where $0\leq k\leq n$. A Johnson Network is a network modeled after Johnson graph $G_i(n,k)$.
The Hamming Scheme $H(n,q,r)$ is an association scheme whose vertex set Q^n is the set of all words of length n over the alphabet Q of q symbols. Two vertices are adjacent if and only if they are r-related, that is, if they differ in exactly r coordinate positions where r is referred to as the Hamming Distance. A Hamming Network is a network modeled after the Hamming graph.
This paper shows that every Johnson Network modeled after $G_i(n,k)$ can also be embedded into the Hamming Network modeled after $H(n,2,2i)$. This is done by showing that there is an embedding of the combined *i*th graphs of the Johnson Scheme $SG_i(n)$, into the Hamming Network $H(n,q,r)$, when q=2 and $r=2i$..

Key-Words: - embeddings, hamming scheme, johnson scheme, association schemes, graph theory, networks

1 Introduction

1.1 On Association Schemes

The theory of association schemes has its roots in the statistical design of experiments and in the study of groups acting on finite sets. In 1973, an important new role for association schemes emerged through the work of Philippe Delsarte [10] and others: certain association schemes were shown to play a central part in the study of error-correcting codes. [1]

Since 1973, the area has been quite active and many advances have been made. Yet researchers continue to approach the subject from a variety of distinct perspectives. Current research in the area includes statistical design of experiments, PBIBD's, finite group actions, character theory, distance-regular graphs, P- and Q-polynomial association schemes, extremal graph theory, knot theory, spin models, Type II matrices, geometries, buildings, coding theory, combinatorial designs etc.

1.2 On Graph Embeddings

An embedding is a representation of a topological object, manifold, graph, field, etc. in a certain space in such a way that its connectivity or algebraic properties are preserved. In graph embedding, connectivity is preserved. [2] An embedding of a source graph G into a host graph is a mapping of the vertices of G into vertices of H and of the edges of G into simple paths of H.[3] Obviously, it would be best and it would definitely satisfy embedding requirements if the source graph is a spanning subgraph of the host graph.

Graph embedding is an important and interesting approach to parallel computing. It is used to model simulation of networks and algorithm structures on different networks. In this process involves assigning subtasks of one network topology to another network topology such that communication overhead is low. Furthermore, if an algorithm is already developed for an architecture using one type of topology, then it can easily be ported to another topology. [7],[8],[9] Embedding may also give useful hints on lower and upper bounds on potential performance of algorithms mapped to parallel machines with static interconnection network topology [3].

1.3 This Study

By embedding the Johnson Networks into the Hamming Network, we open the possibility that certain algorithms developed for the Johnson Network may also work the Hamming Network. In terms of association schemes, showing such embedding establishes a pattern on the "behavior" of the two association schemes, i.e. when their *i*th associates coincide, thus showing a relationship between the two schemes. This is done by showing that each Johnson Graph $G_i(n,k)$ is a subgraph of after $H(n,2,2i)$.

2 Definitions

2.1 Johnson Scheme and the Johnson Graphs

Let n and k be fixed positive integers. The **Johnson Scheme** $\boldsymbol{G(n,k)}$ is an association scheme whose vertex set is the set of all k-subsets of a fixed set of n elements. Two vertices A and B are i-related if $|A \cap B| = k\text{-}i$, and i is referred to as the **johnson distance.** This scheme has k classes. [4]

The **Johnson Graph** $\boldsymbol{G(n,k)}$ or the graph of the Johnson Scheme of the first order is the undirected graph where the vertices are all the k-subsets of a fixed n-set. Two vertices A and B are adjacent if and only if $|A \cap B| = k\text{-}1$ [5]. The order of $G_i(n,k)$ is ${}^nC_k$ and that each vertex is $k(n\text{-}k)$ regular.

The ***i*th Johnson Graph** $\boldsymbol{G_i(n,k)}$ is the undirected graph where the vertices are also all the k-subsets of a fixed n-set. Here two vertices A and B are adjacent if and only if $|A \cap B| = k\text{-}i$. A **Johnson Network** is a network modeled after Johnson graph $G_i(n,k)$.

The **Combined *i*th Johnson Graphs** $\boldsymbol{SG_i(n)}$ or the ***i*th Johnson Networks** is the graph formed by getting the graph sum of all the $G_i(n,k)$ graphs. Thus

$$SG_i(n) \text{ is } \bigcup_{k=1}^{n} G_i(n,k) \tag{1}$$

$V(SG_i(n))$ will be equal to $V(G_i(n,0)) \cup V(G_i(n,1)) \cup V(G_i(n,2)) \cup \ldots \cup V(G_i(n,n))$.

Furthermore, since $n(V(G_i(n,k))) = {}^nC_k$ then $n(V(SG_i(n)))$ is equal to

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n. \tag{2}$$

and $V(SG_i(n))$ *is* set of the 2^n subsets of the set $\{1,2,3,\ldots,n\}$

Thus $SG_1(2)$ is the graph formed by combining the graphs $G_i(2,0)$, $G_i(2,1)$ & $G_i(2,2)$ as we see in Figure 1. The vertex set $V(SG_1(2))$ is set of the $2^2=4$ subsets of the set $\{1,2\}$, namely $\{\}$, $\{1\}$, $\{2\}$ and $\{1,2\}$

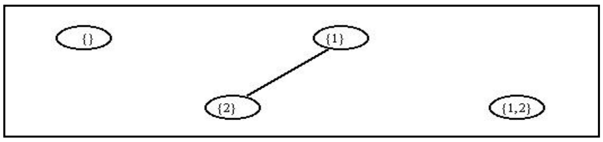

Figure 1 $SG_i(2)$ is the graph formed by combining the graphs $G_1(2,0)$, $G_1(2,1)$ & $G_1(2,2)$

2.2 Hamming Scheme and Hamming Graphs

Let Q be an alphabet of q symbols. The Hamming Scheme H(n,q) is an association scheme whose vertex set Q^n, the set of all words of length n over Q. Two words are r-related if they differ in exactly r coordinate positions, where r is referred to as the **hamming distance.** In the **Hamming Graph** $\boldsymbol{H(n,q,r)}$, the vertices are adjacent if and only if they are r-related. This scheme has n classes. [4] Thus $|V(H(n,q,r)|=q^n$.

A **Hamming Network** is a network modeled after the Hamming graph.

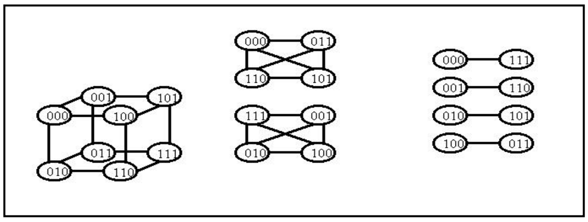

Figure 2 The $H(3,2,m)$ – cubes, m = 1,2,3

The *k*-subgraph of the Hamming Graph $\boldsymbol{H_k(n,2,r)}$ is the induced subgraph of the Hamming Graph $H(n,q,r)$ where q=2, such a vertex $v \in V(H_k(n,2,r))$ if and only if $v \in V(H(n,2,r))$ and v is of parity k.

Since k has n possible values, then $H(n,2,r)$ has n k-subgraphs $H_k(n,2,r)$. The number of all strings of length n that are of parity k is just as same as taking k elements from a set of size n. Thus $n(V(H_k(n,2,r))) = {}^nC_k$

The **Parity Subgraph of the Hamming Graph** $\boldsymbol{H_S(n,2,r)}$ is the spanning subgraph of the Hamming Graph $H(n,2,r)$ such that vertices u and v are adjacent if and only if they are of the same parity. $\boldsymbol{H_S(n,2,r)}$ is the graph formed by getting the graph sum of all the k-subgraphs of the Hamming Graph $H_k(n,2,r)$.

$$H_S(n,q,r) \text{ is } \bigcup_{k=1}^{n} H_k(n,q,r) \tag{3}$$

Furthermore, since $n(V(H_k(n,2,r))) = {}^nC_k$ then $n(V(H_S(n,2,r)))$ is equal to

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n. \quad (4)$$

Since $H_S(n,2,r)$ is a spanning subgraph of $H(n,2,r)$, then $V(H_S(n,2,r)) = V(H(n,2,r))$. However, the two graphs are still not isomorphic because there exists edges $uv \in E(H(n,2,r))$ where $uv \notin E(H_S(n,2,r))$ since u and v are not of the same parity.

3 The Embedding

The outline of our strategy for showing this embedding is as follows:

- Show $G_i(n,k)$ is isomorphic to $H_k(n,2,2i)$
- Show $SG_i(n)$ is isomorphic to $H_S(n,2,2i)$
- Conclude that $SG_i(n)$ is a spanning subgraph of $H(n,2,2i)$
- Conclude that there is an embedding of $SG_i(n)$ into $H(n,2,2i)$

Lemma 1 $G_i(n,k)$ is isomorphic to $H_k(n,2,2i)$

Proof:
We do this by first introducing a mapping P: $G_i(n,k) \rightarrow H_k(n,2,2i)$. Thus, we have to show that there is a mapping $P(v)=v'$, of $v = a_1 a_2 a_3 ... a_n \in V(H_k(n,2,2i))$ into $v' \in V(G_i(n,k))$ such that $a_j = 1$ if and only if $j \in v'$. We will then show that this mapping preserves adjacency.

Let $H(n,2,m)$ by a Hamming Graph and $V(H(n,2,m))$ be the set of 2^n vertices which are binary numbers. Each vertex v is an n-tuple $\{a_1 a_2 a_3 ... a_n\}$ where a_j is either 0 or 1. Let $H_k(n,2,m)$ by the kth subgraph of the Hamming Graph, thus $v \in V(H_k(n,2,m))$ if there are k bits $a_j = 1$ for $1 \leq j \leq n$. If we define a set Sv_i for each vertex in $v_i \in V(H_k(n,2,m))$, where $1 \leq i \leq {}^nC_k$ and $j \in Sv_i$ if $a_j = 1$ in v_i. Thus, we would be obtaining a total of ${}^nC_k$ sets Sv_i each of which is of size k. Furthermore, each set Sv_i will resemble exactly one vertex v' in $G_i(n,k)$. Thus, there will be a one-is-to-one onto correspondence between a vertex $v \in V(H_k(n,2,m))$ and a vertex $v' \in G_i(n,k)$

Hence, there is a mapping $P(v)=v'$, of $v = a_1 a_2 a_3 ... a_n \in V(H_k(n,2,m))$ into and $v' \in V(G_i(n,k))$ such that $a_j = 1$ if and only if $j \in v'$ □

For instance, if n=3 then $H(3,2,m)$ is a Hamming Graph and $V(H(3,2,m))$ be the set of $2^3 = 8$ vertices: 000, 001, 010, 011, 100, 101, 110 and 111.

Furthermore, for $k = 0$ to n

$V(H_0(3,2,m)) = \{000\}$
$V(H_1(3,2,m)) = \{001, 010, 100\}$
$V(H_2(3,2,m)) = \{101, 110, 011\}$ and
$V(H_3(3,2,m)) = \{111\}$

Also, if we get the all vertices of $G_i(3,k)$, for $k = 0$ to n, we would be having the set of the $2^3 = 8$ subsets of the set $\{1,2,3\}$ namely:

$V(G_i(3,0))=\{\}$
$V(G_i(3,1))=\{1\},\{2\},\{3\}$
$V(G_i(3,2))=\{1,2\},\{2,3\}, \{1,3\}$ and
$V(G_i(3,3))=\{1,2,3\}$

P is the mapping $P(v)=v'$ yielding the values in Table 1

Table 1. The Mapping $P(v)=v'$ for k = 0 to 3

k	*v*	*v'*
0	000	{}
1	001	{3}
	010	{2}
	100	{1}
2	011	{2,3}
	101	{1,3}
	110	{1,2}
3	111	{1,2,3}

Now, we will have to show that this mapping preserves adjacency and non-adjacency. Note that two vertices v'_x and $v'_y \in V(G_i(n.k))$ are adjacent if and only if they have the same number of elements k and $v'_x \cap v'_y = k-i$. Furthermore, there are i elements in v'_x that are not in v'_y and vice versa. See Figure 3

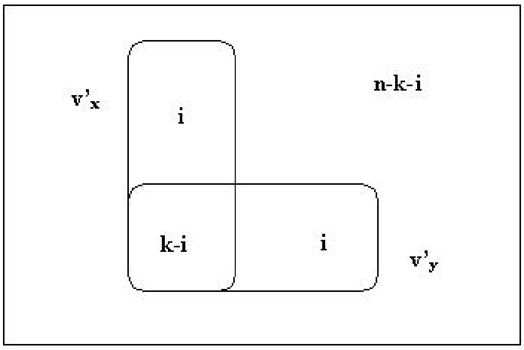

Figure 3 The adjacent vertices v'_x and v'_y in $G_i(n,k)$

This also means that v_x and $v_y \in V(H_k(n,2,2i))$ where v_x and v_y each have k bits = 1. According to definition, two vertices in $H_k(n,2,m)$ are adjacent if and only if they are different in m bits.

Note also that if vertices v'_x and v'_y are adjacent in $G_i(n,k)$, their corresponding vertices v_x and v_y in

$H_k(n,2,2i)$ are also adjacent if and only if they are different in $2i$ bits. For a vertex $v'_x \in G_i(n,k)$ to be adjacent to another vertex $v'_y \in G_i(n,k)$, $|v'_x - v'_y| = |v'_y - v'_x| = i$.

Since $|v'_x - v'_y| = i$, there has to be i bits in $v_x = 1$ but are 0 in v_y. And since $|v'_y - v'_x| = i$, there are another i bits $v_y = 1$ but are 0 in v_x. Hence, v_x and v_y are different in a total of $2i$ bits.

In Figure 4 shows us two vertices $v_y = 001110$ and $v_x = 110100$. They both have k bits = 1 but there i bits (i=2) in $v_x = 1$ but are 0 in v_y and there are another i bits $v_y = 1$ but are 0 in v_x. They are different in k-i bits where k-1 = 1. And they are different in a total of $2i = 4$ bits.

They have corresponding vertices $v'_x = \{1,2,4\}$ and $v'_x = \{4,5,6\}$ in V($SG_i(n)$). They have the same number of elements say k=3 and $v'_x \cap v'_y = k-i = 1$. There are i=2 elements in v'_x that are not elements of v'_y and vice versa.

Hence two adjacent vertices v'_x and $v'_y \in$ V($SG_i(n)$) have corresponding vertices v_x and $v_y \in$ V($H_k(n,2,m)$) that are adjacent if and only if $m = 2i$.

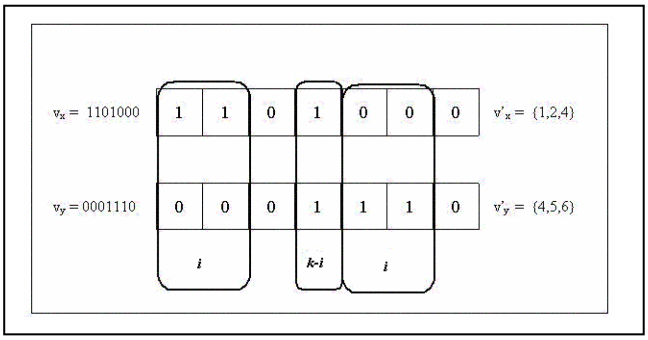

Figure 4 Vertices adjacent in both $G_i(n,k)$ and $H_k(n,2,2i)$

Now since all the vertices in $G_i(n,k)$ have corresponding vertices in $H_k(n,2,2i)$ and all edges in $G_i(n,k)$ have corresponding edges in $H_k(n,2,2i)$, and vice versa, then the ith graphs of the Johnson Scheme $G_i(n,k)$ is isomorphic to the kth subgraph of the Hamming Scheme $H_k(n,2,2i)$.

Lemma 2 $SG_i(n)$ is isomorphic to $H_S(n,2,2i)$

Proof:

Note that $H_S(n,2,2i)$ is graph formed by getting the graph sum of all the k-subgraph of the Hamming Graph $H_k(n,2,r)$, while $SG_i(n)$ is the graph formed by getting the graph sum of all the $G_i(n,k)$ graphs. Furthermore, n(V($H_S(n,2,2i)$)) and n(V($SG_i(n)$)) are both equal to 2^n. Also the mapping $P(v)=v'$ still holds for this case where $v = a_1a_2a_3...a_n \in$ V($H_S(n,2,m)$) and v'$\in$ V($SG_i(n)$) such that $a_j = 1$ if and only if $j \in v'$. Note also that since $SG_i(n)$ and $H_S(n,2,2i)$ are both graph sums, then they will just contain the union of all the edges of their respective component graphs and no edges will be added or subtracted. Thus the adjacency of vertices are preserved.

Hence, $SG_i(n)$ is isomorphic to $H_S(n,2,2i)$ □

Now, since $H_S(n,2,2i)$ is a spanning subgraph of $H(n,2,2i)$, then $SG_i(n)$ is also a spanning subgraph of $H(n,2,2i)$.

Theorem 1 There is an Embedding of the combined ith graphs of the Johnson Scheme $SG_i(n)$ into the graph of the Hamming Scheme $H(n,2,2i)$

Proof:

Since according to Lemma 2, $SG_i(n)$ is also a spanning subgraph of $H(n,2,2i)$, we can also conclude that there is an embedding of the ith graphs of the Johnson Scheme $SG_i(n)$ into the graph of the Hamming Scheme $H(n,2,2i)$. Since the source graph is also a subgraph, this is an embedding where load, congestion, dilation and expansion are all equal to 1. □

Moreover, since $SG_i(n)$ is defined as the graph sum of all ith Johnson Graphs $G_i(n,k)$ graphs, these also hold:

Corollary 1 Every ith Johnson Graph $G_i(n,k)$ is a subgraph of the Hamming Graph $H(n,2,2i)$.

Corollary 2 Every Johnson Network $G_i(n,k)$ can be embedded into the Hamming Network $H(n,2,2i)$

In terms of associations schemes, the instance when q=2 and r=2i is one wherein two vertices v'_x and v'_y that are i-related in the Johnson Scheme $G_i(n,k)$ have their counterpart vertices v_x and v_y in the Hamming Scheme $H(n,q,r)$ also r-related. In other words, the combined graphs of the Johnson Scheme embeds the graph of the Hamming Scheme when q=2 and the Hamming Distance is twice the Johnson Distance. Thus Johnson Networks modeled after $G_i(n,k)$ can be embedded into Hamming Networks modeled after $H(n,2,2i)$.

The following figures show embeddings of $G_i(n,k)$ into $H(n,2,r)$ where the alphabet $Q = \{0,1\}$. As mentioned, $SG_i(n)$ and $H(n,2,r)$ have corresponding nodes. The graph $H(n,2,r)$ is shown by the black edges and the binary labels. The graphs $SG_i(n)$, and $H_S(n,2,2i)$, are shown here with colored edges, with a particular color assigned for each $G_i(n,k)$ and $H_k(n,2,2i)$. The vertices of $G_i(n,k)$ are labeled as sets and are also colored accordingly.

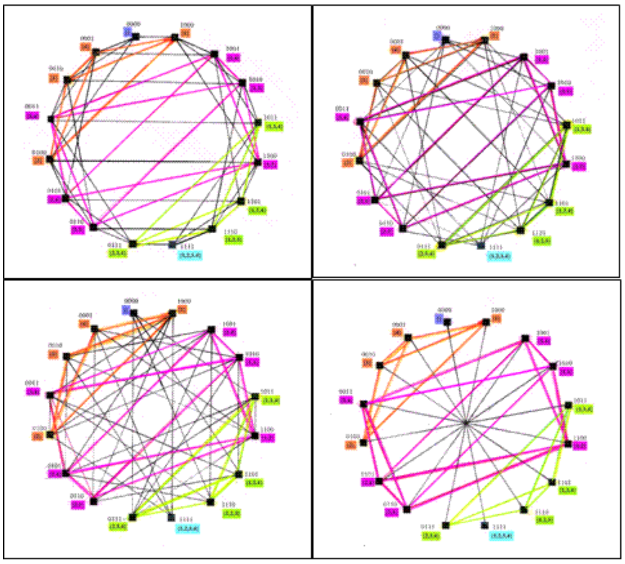

Figure 5 $SG_1(4)$ with $H(4,2,1)$ (upper left), with $H(4,2,2)$ (upper right), with $H(4,2,3)$ (lower left) and with $H(4,2,4)$ (lower right)

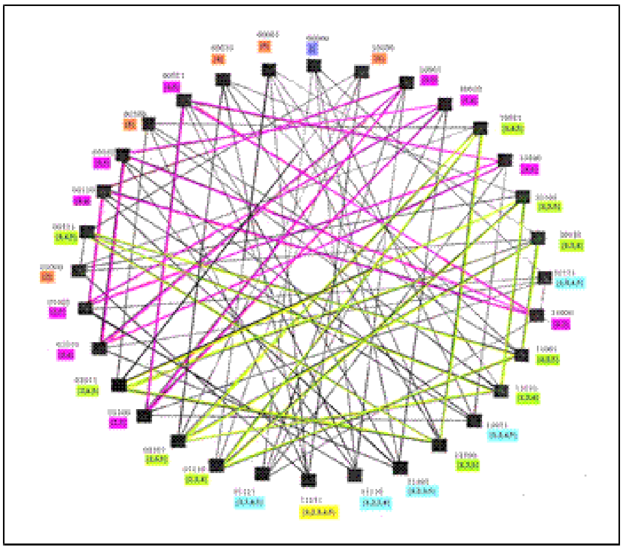

Figure 6 $SG_2(5)$ with $H(5,2,4)$

4 Conclusion

By embedding the Johnson Networks into the Hamming Network, it can be concluded that certain algorithms developed for a network whose architecture is similar to that of the combined *i*th Johnson Networks may also be implemented for another network whose architecture is similar to that of the Hamming Network.

Furthermore, it may be concluded that such algorithms may also be implemented into networks whose architectures are isomorphic to the Hamming Network.

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