Conversion Of First Order Linear Vector Differential Equations With Polynomial Coefficient Matrix To Okubo Form

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Abstract: - One can use power series expansions in the solution of the first order linear vector differential equations as long as the expansion is realized around a regular point of the differential equation. However the utilizability and the practicality of the expansion depends on the structure of the recursion amongst the coefficents of the expansion and the most preferable case uses a first order (two term) recursion. If the coefficient matrix of the equation is a polynomial then the recursion between the certain consecutive coefficients remains finite but its order is generally higher than one. Although there are various tools to handle this situation it is better to change the structure of the equation by defining new unknowns and to increase its vector dimension appropriately to get a new first order linear vector differential equation with a matrix coefficient which takes us to first order recursion amongst the expansion coefficients.

Key-Words: - Linear Vector Differential Equations, Solution Space Extension, Okubo Form, Series Solutions

1 Introduction

The most general form of a first order linear homogeneous vector differential equation can be given as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t)$ stands for a vector valued function with n elements and $\mathbf{A}(t)$ denotes an $n \times n$ matrix with t dependent elements where t symbolizes the independent variable which is considered as time in many applications. Here we assume that this variable can take also complex values. This assumption is necessary for the convergence discussions. The differential equation in (1) may have regular or irregular singular points in the complex plane of t depending on the structure of $\mathbf{A}(t)$. Its solution at a regular point in t complex plane is a Taylor series and converges in the complex planar disk centered at the expansion point and having a radius joining the center to the closest singular point excluding the boundary.

The solutions at regular singular points may have branch cuts and/or logarithmic singularities depending on the structure of $\mathbf{A}(t)$ and they converge inside the complex planar disk centered at the singularity and having the closest singularity at its boundary. In this case the center of the disk and its boundary may have to be discarded from the convergence region.

If the expansion point is an irregular singularity then the solution is an asymptotic expansion around the expansion point and diverges everywhere except the expansion point.

Since our purpose here is just to show how to convert (1) into Okubo's more amenable form we avoid to deal with the regular or irregular singularities. They can be tackled with after gathering sufficient information about and gaining experience on this task here. Hence we assume that $\mathbf{A}(t)$ can be expanded into a Taylor series at any regular point of (1) in the complex plane of t. Since it is easier to deal with the polynomials than the tackling with the infinite series even if they converge everywhere. Hence,

$$\mathbf{A}(t) \equiv \sum_{j=0}^{m} t^j \mathbf{A}_j \tag{2}$$

where $\mathbf{A}_j$, $(0 \le j \le m)$ stand for the given, constant, $n \times n$ type matrices. The matrix $\mathbf{A}(t)$ given in (2) is analytic everywhere in the complex plane of t except infinity. Hence, with this matrix, the differential equation in (1) has no singular point in t complex plane. Therefore the solution for this equation must be a usual Taylor series expanded around a given point of t–plane. To simplify the investigations without losing any generality we assume that the regular point around which the solution will be expanded into Taylor series is the origin of the t complex plane. Therefore we assume the following form for the solution of (1) under the definition of (2)

$$\mathbf{x}(t) \equiv \sum_{j=0}^{\infty} t^j \mathbf{x}_k \tag{3}$$

where $\mathbf{x}_j$, $(0 \leq j < \infty)$ stand for the constant expansion coefficients which are unknown yet. To evaluate them we can use (3) and (2) in (1) and get the following recursion

$$(j+1)\mathbf{x}_{j+1} = \sum_{k=0}^{j} \mathbf{A}_{j-k}\mathbf{x}_k, \qquad 0 \leq j < \infty \tag{4}$$

where $\mathbf{A}_j \equiv 0$ if $j > m$. This recursion can be divided into two sets of equations

$$(j+1)\mathbf{x}_{j+1} = \sum_{k=j-n}^{j} \mathbf{A}_{j-k}\mathbf{x}_k, \qquad m \leq j < \infty \tag{5}$$

and

$$(j+1)\mathbf{x}_{j+1} = \sum_{k=0}^{j} \mathbf{A}_{j-k}\mathbf{x}_k, \qquad 0 \leq j < m \tag{6}$$

where (5) defines an $(m+1)$–th order difference equation (recursion) while (6) defines the initial conditions for (5). There remains $(m+1)$ unknown $\mathbf{x}$ values, $\mathbf{x}_0$,...,$\mathbf{x}_m$, in (5) and those values are reduced into just a single arbitrary vector, $\mathbf{x}_0$, through the equations of (6).

The recursion in (5) would be a nonlocal (order depends on the index paramater of the recursion) difference equation if $\mathbf{A}(t)$ were an infinite series in t instead of a polynomial. In fact, this was the main reason why we have assumed the polynomial structure. However we should emphasize on the fact that the nonlocal recursions can be handled by using infinite dimensional vectors and matrices in the conversions to Okubo form.

The $(m+1)$–th order recursion in (5) can be converted to a first order recursion with higher dimensional vectors through order reducing methods. We do not prefer these approches. Instead, we use an appropriate space extension approach for converting the equation (1) with the matrix coefficient given by (2). The space extension is somehow a partitioning of the Taylor series into subseries in powers of a new independent variable which is in fact an integer power of the present independent variable such that each subseries is multiplied by a different integer power of the original independent variable.

The paper is organized as follows. The second section contains the space extension[1-9] approach to reduce the equation under consideration to Okubo's form. To this end, we give all important details and especially correspondences. The third section involves the construction of the Okubo form. We give basic ideas to this end although explicit expressions of certain final entities are not given. The solution of Okubo form is also not given. They will be focused on in a different paper. The fourth section briefly covers the concluding remarks.

2 Space Extension

We define a new independent variable as follows

$$\theta \equiv t^{m+1} \tag{7}$$

and then write

$$\mathbf{x}(t) \equiv \boldsymbol{\xi}_1(\theta) + t\boldsymbol{\xi}_2(\theta) + \cdots + t^m\boldsymbol{\xi}_{m+1}(\theta) \tag{8}$$

This equation defines a polynomial in t with vector coefficients if we forget the dependence of θ on t. Hence we can consider the coefficients of this polynomial as the block components of a vector. We write

$$\boldsymbol{\xi}(\theta) \equiv [\, \xi_1(\theta) \ \ldots \ \xi_{m+1}(\theta) \,] \tag{9}$$

Therefore $\boldsymbol{\xi}(\theta)$ will be replaced with $\mathbf{x}(t)$ in our further formulations. However, we need to establish a rule for the differentiation with respect to t. This rule will contain the differentiation with respect to θ. We also need the multiplication of $\boldsymbol{\xi}(\theta)$ with a matrix polynomial in t.

Although we can take the first derivative of the both sides of (8) for differentiation rule it is better to deal with the following form instead of (8).

$$\mathbf{x}(t) \equiv \left[\sum_{j=1}^{m+1} t^{j-1} \left(\mathbf{e}_j \otimes \mathbf{I}_n\right)^T \right] \boldsymbol{\xi}(\theta) \tag{10}$$

where $\mathbf{e}_j$, $(1 \leq j \leq m)$, stands for the cartesian unit vector whose all elements except the j–th one which is 1 vanish and $\otimes$ symbolizes the direct matrix product which is defined such that each element of its left operand is multiplied by its right operand and the resulting entity is replaced with the element under consideration. Here $\mathbf{I}_n$ represents the $n \times n$ identity matrix.

Now the differentiation of the both sides of (10) by keeping the t dependence of θ in mind gives the following equation after some reorganizations to get read of t's explicit powers higher than m

$$\begin{aligned} \dot{\mathbf{x}}(t) \ = \ & \sum_{j=1}^{m} j t^{j-1} \overline{\mathbf{e}}_{j+1}^T \boldsymbol{\xi}(\theta) + t^m \overline{\mathbf{e}}_1^T \dot{\boldsymbol{\xi}}(\theta) \\ & + \sum_{j=1}^{m} t^{j-1} \overline{\mathbf{e}}_{j+1} \theta \dot{\boldsymbol{\xi}}(\theta) \end{aligned} \tag{11}$$

where $\dot{\boldsymbol{\xi}}(\theta)$ stands for the derivative of $\boldsymbol{\xi}$ with respect to θ and

$$\bar{\mathbf{e}}_j \equiv \mathbf{e}_j \otimes \mathbf{I}_n, \qquad 1 \le j \le m+1 \tag{12}$$

Now we can write the following vector differentiation rule which has no explicit t dependence.

$$\begin{aligned} \frac{d\boldsymbol{\xi}(\theta)}{dt} &= \left[\sum_{j=1}^{m} j\bar{\mathbf{e}}_j\bar{\mathbf{e}}_{j+1}^T\right]\boldsymbol{\xi}(\theta) + \bar{\mathbf{e}}_{m+1}\bar{\mathbf{e}}_1^T\dot{\boldsymbol{\xi}}(\theta) \\ &\quad + \left[\sum_{j=1}^{m} \bar{\mathbf{e}}_j\bar{\mathbf{e}}_{j+1}^T\right]\theta\dot{\boldsymbol{\xi}}(\theta) \end{aligned} \tag{13}$$

If we define

$$\begin{aligned} \mathbf{D}_0 &\equiv \sum_{j=1}^{m} j\bar{\mathbf{e}}_j\bar{\mathbf{e}}_{j+1}^T \\ \mathbf{D}_1(\theta) &\equiv \theta\sum_{j=1}^{m} \bar{\mathbf{e}}_j\bar{\mathbf{e}}_{j+1}^T + \bar{\mathbf{e}}_{m+1}\bar{\mathbf{e}}_1^T \end{aligned} \tag{14}$$

then (13) can be written as follows

$$\begin{aligned} \frac{d\boldsymbol{\xi}(\theta)}{dt} &= \mathbf{D}_0\boldsymbol{\xi}(\theta) + \mathbf{D}_1(\theta)\dot{\boldsymbol{\xi}}(\theta) \\ &= \mathcal{D}(\theta)\boldsymbol{\xi}(\theta) \end{aligned} \tag{15}$$

where apparently

$$\mathcal{D}(\theta) \equiv \mathbf{D}_0 + \mathbf{D}_1(\theta)\frac{d}{d\theta} \tag{16}$$

This equation means that the differentiation with respect to t in the original vector space (that is $\dot{\mathbf{x}}(t)$) is corresponded by the operator $\mathcal{D}(\theta)$'s action in the extended space (that is $\mathcal{D}(\theta)\boldsymbol{\xi}(\theta)$). This is the differentiation correspondence and we also need the correspondence for multiplication by a matrix.

The rule for the multiplication with the coefficient matrix can be constructed in a similar way. To this end we need to construct an extended matrix notation acting on $\boldsymbol{\xi}(\theta)$. For this purpose we consider the matrix $\mathbf{A}(t)$ in (2) as an infinite sum under the constraint $\mathbf{A}_j = 0, \quad j \ge m$ and the vector $\mathbf{x}(t)$ in (8) as an infinite sum under the constraint $\boldsymbol{\xi}_j = 0, \quad j \ge m+1$. This enables us to use the Cauchy Product Formula for infinite series and to write

$$\mathbf{A}(t)\mathbf{x}(t) = \sum_{j=0}^{\infty} t^j \sum_{k=0}^{j} \mathbf{A}_{j-k}\boldsymbol{\xi}_{k+1}(\theta) \tag{17}$$

which turns out to be

$$\begin{aligned} \mathbf{A}(t)\mathbf{x}(t) &= \sum_{j=0}^{2m} t^j \sum_{k=0}^{j} \mathbf{A}_{j-k}\boldsymbol{\xi}_{k+1}(\theta) \\ &= \sum_{j=0}^{m} t^j \sum_{k=0}^{j} \mathbf{A}_{j-k}\boldsymbol{\xi}_{k+1}(\theta) \\ &\quad + \sum_{j=m+1}^{2m} t^j \sum_{k=0}^{j} \mathbf{A}_{j-k}\boldsymbol{\xi}_{k+1}(\theta) \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\sum_{j=m+1}^{2m} t^j \sum_{k=0}^{j} \mathbf{A}_{j-k}\boldsymbol{\xi}_{k+1}(\theta) \\ &= \sum_{j=0}^{m-1} t^j \sum_{k=0}^{j+m+1} \mathbf{A}_{j+m+1-k}\theta\boldsymbol{\xi}_{k+1}(\theta) \\ &= \sum_{j=0}^{m-1} t^j \sum_{k=0}^{m} \mathbf{A}_{j+m+1-k}\theta\boldsymbol{\xi}_{k+1}(\theta) \\ &= \sum_{j=0}^{m-1} t^j \sum_{k=j+1}^{m} \mathbf{A}_{j+m+1-k}\theta\boldsymbol{\xi}_{k+1}(\theta) \\ &= \sum_{j=0}^{m} t^j \sum_{k=j+1}^{m} \mathbf{A}_{j+m+1-k}\theta\boldsymbol{\xi}_{k+1}(\theta) \end{aligned} \tag{19}$$

These results enable us to write

$$\mathbf{A}(t)\mathbf{x}(t) = \sum_{j=1}^{m+1} t^{j-1} \sum_{k=1}^{m+1} \mathcal{A}_{jk}(\theta)\boldsymbol{\xi}_k(\theta) \tag{20}$$

where

$$\mathcal{A}_{jk}(\theta) = \begin{cases} \mathbf{A}_{j-k} & 1 \le k \le j \\ \theta\mathbf{A}_{j-k+m+1} & j+1 \le k \le m+1 \end{cases} \tag{21}$$

If we define

$$\mathbf{t}^T \equiv [\,\mathbf{I}_n \quad t\,\mathbf{I}_n \quad \ldots \quad t^m\,\mathbf{I}_n\,] \tag{22}$$

then we can write

$$\mathbf{A}(t)\mathbf{x}(t) = \mathbf{t}^T\boldsymbol{\mathcal{A}}(\theta)\boldsymbol{\xi}(\theta) \tag{23}$$

Therefore we conclude that the image of $\mathbf{x}(t)$ under $\mathbf{A}(t)$ is equivalent the image of $\boldsymbol{\xi}(\theta)$ under $\boldsymbol{\mathcal{A}}(\theta)$ in the extended space.

The equations (15) and (23) can be formally rewritten as follows to get the correspondences for the input and output spaces of the space extension

$$\frac{d}{dt} \Longrightarrow \mathcal{D}(\theta), \qquad \mathbf{A}(t) \Longrightarrow \mathcal{A}(\theta) \tag{24}$$

The equations in (24) are the fundamental correspondence rules for the **Space Extension**. We will use them to get Okubo form.

3 Okubo Form

Now the correspondences in (24) enables us to write

$$\mathcal{D}(\theta)\boldsymbol{\xi}(\theta) = \mathcal{A}(\theta)\boldsymbol{\xi}(\theta) \tag{25}$$

and a careful investigation shows that

$$\left[\sum_{j=1}^{m} \overline{\mathbf{e}}_{j+1}\overline{\mathbf{e}}_j^T + \theta\overline{\mathbf{e}}_1\overline{\mathbf{e}}_{m+1}^T\right]\mathcal{D}_1(\theta) = \theta\left(\mathbf{I}_m \otimes \mathbf{I}_n\right) \tag{26}$$

$$\left[\sum_{j=1}^{m} \overline{\mathbf{e}}_{j+1}\overline{\mathbf{e}}_j^T + \theta\overline{\mathbf{e}}_1\overline{\mathbf{e}}_{m+1}^T\right]\mathcal{D}_0 = \sum_{j=1}^{m} j\overline{\mathbf{e}}_{j+1}\overline{\mathbf{e}}_{j+1}^T \tag{27}$$

$$\left[\sum_{j=1}^{m} \overline{\mathbf{e}}_{j+1}\overline{\mathbf{e}}_j^T + \theta\overline{\mathbf{e}}_1\overline{\mathbf{e}}_{m+1}^T\right]\mathcal{A}(\theta) - \sum_{j=1}^{m} j\overline{\mathbf{e}}_{j+1}\overline{\mathbf{e}}_{j+1}^T = \mathbf{B}_0 + \theta\mathbf{B}_1 \tag{28}$$

where $\mathbf{B}_0$ and $\mathbf{B}_1$ are certain constant matrices. All these definitions urges us to write the ultimate Okubo form as follows

$$\frac{d\boldsymbol{\xi}(\theta)}{d\theta} = \left[\frac{1}{\theta}\mathbf{B}_0 + \mathbf{B}_1\right]\boldsymbol{\xi}(\theta) \tag{29}$$

This equation has a regular singularity at $\theta = 0$ and the asymptotic solution around that point can be written as

$$\lim_{\theta\to 0}\boldsymbol{\xi}(\theta) = \theta^{\mathbf{B}_0}\boldsymbol{\xi}_0 \tag{30}$$

where $\boldsymbol{\xi}_0$ is an arbitrary constant vector. This asymptotic form implies that the singularity of the solution at $\theta = 0$ is determined completely by the eigenvalues of the matrix $\mathbf{B}_0$. We are not going to prove anything about this spectrum here. However, it is quite natural to expect eigenvalues 0, $\frac{1}{m+1}$, ..., $\frac{m}{m+1}$ to reflect the branching nature in the definition of θ in terms of t.

(29) has an irregular singularity at $\theta = \infty$ and the asymptotic solution around that point can be written as

$$\lim_{\theta\to\infty}\boldsymbol{\xi}(\theta) = e^{\theta\mathbf{B}_1}\boldsymbol{\xi}_\infty \tag{31}$$

where $\boldsymbol{\xi}_\infty$ is an arbitrary constant vector. This asymptotic form implies that the singularity of the solution at $\theta = \infty$ is determined completely by the eigenvalues of the matrix $\mathbf{B}_1$. We do not intend to deal with the asymptotic solution around this point.

4 Concluding Remarks

The main goal of this paper has been to convert a linear first order ordinary vector differential equation to Okubo form via a space extension. The basic focus here has been on the space extension. Although we do not give the series solution of the Okubo form it is quite straightforward to get them by using standard methods of series expansion. We do not give these details although they will be included during the presentation in the conference.

Here the space extension is realized around a regular point of the original differential equation which is assumed not to have singularities in finite domains of t complex plane. The cases where regular or irregular singular points exist have been left to future investigations. We also have not been focused on any asymptotic behavior here.

Acknowledgement:
Author is grateful to Turkish Academy of Sciences for its invaluable support.

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