New Geometric Constructions to Determine the Radius of Curvature of Conics at any Point

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Abstract: - Compass-and-straightedge constructions have a long tradition and still are of very genuine mathematical interest [1]. Some recent results in the geometry of conics with remarkable constructive and engineering applications [2] also suggest original constructions to obtain the radius of curvature of conics at any given point using compass and straightedge alone. Three versions of the method are presented with figures, descriptions of constructive steps, and justifications of their exactitude to solve the problem stated.

Key-Words: - Compass-and-Straightedge Constructions; Geometry of Conics; Ellipse; Hyperbola; Parabola; Radius of Curvature.

1 Introduction

Straightedge and compass geometric constructions are classic, dating back to the ancient Greeks; their interest is certainly historical, but also geometrical and mathematical in a very basic sense [1]. As is well-known, the operations allowed in these constructions are only two: to draw a straight line joining two given points, and to draw a circle whose center and radius are known. Each point must then be determined as the intersection of two straight lines, or a line and a circle, or two circles.

Conic sections are among the most intensively applied curves in Computational Geometry, Computer Graphics and other fields related with Geometry and Geometric Design, and the radius of curvature is a crucial differential property of any curve in the above-mentioned fields. This, together with the relevance of straightedge-and-compass constructions, makes it worthy to study any new construction finding the radius of curvature of a conic at any given point by means of straightedge and compass alone.

Bibliography presenting a solution to such an obvious problem of a classic type is not as common as one might expect. Classics in the field like [3] or [4] offer no direct account of it (although it may be possible to infer a solution based on the principles discussed therein). A method to solve the problem for an ellipse was published in [5]; it has the slight disadvantage that it does not work with its vertices. In [6] the same method is shown, but two alternative constructions are provided to solve the vertices case. Such alternatives, in contrast, are not necessary with the constructions presented in this paper.

The approach proposed here is different and original. It is based on a recently found property of conic sections, namely, that *the cube root of the radius of curvature of a conic at any of its points is directly proportional to the length of the segment perpendicular to the conic from that point to any one of its main axes*.

It may be surprising that such simple new insights on conics can still come to light nowadays. Proof and discussion of this property, along with consequences for the generation of families of surfaces of conic sections with many interesting applications, can be found in [2]. More specifically, a particular case of these families of surfaces makes them ruled, which greatly facilitates their constructive applications (even if they do not turn out to be developable, they can be realized by means of straight beams, tie rods, etc., and indeed many applications have already been carried out in Civil Engineering, Art and Architecture, Geometric Design, etc.)

In this paper, that same property of conics inspires some different, but related, geometric constructions. For each one of them, the figure and a description of the successive steps required to draw it are followed by a mathematical justification of its exactitude to solve the straightedge-and-compass problem.

For the elliptical case, the data are half-axes a, b of ellipse E and its point P, in which the radius of curvature R is sought, as can be seen in Fig.1. It has been drawn with $a > b$ (specifically, $a/b = 3/2$), although the case $b > a$ can be drawn analogously, presenting essentially no difference. The only part of the figure that cannot be actually drawn with straightedge and compass is precisely the ellipse *E*; nevertheless, *E* has also been represented to help visualization, and it is possible –and easy– to obtain any number of arbitrary points *P* of *E* with straightedge and compass. Fig.3 below shows a well-known way to do it: after drawing the main circle c_a and the minor circle c_b of *E* (only a quarter of which are represented in the figure), draw any straight line from the origin and determine the respective points of intersection P_1 and P_2. The horizontal line by P_2 and the vertical one by P_1 intersect in *P*, which is a point of the ellipse, as is proved by its parametric equations $\{x = a\cos t,\, y = b\sin t\}$, where the parameter *t* is the angle of the straight line by *O*, P_2 and P_1 with the axis of abscissas (incidentally, $t = 1$ rad in Fig.3).

2 Justified Constructions

In this section, all constructions are fully explained, including figures, with justified constructive steps. A first construction for the elliptical case will be followed by a solution to the hyperbolic one. The parabolic case is similar to them; instead of showing its solution explicitly, a more interesting, optimized version for the case of an ellipse will be presented.

2.1 First Construction for the Elliptical Case

Fig.1 shows the first original construction we present to solve the problem for the elliptical case.

2.1.1 Description of the Constructive Steps

First, the positions of foci *F* and –*F* of *E* are determined (a construction that has not been represented, but is trivial taking into account that $F^2 + b^2 = a^2$). Both foci are joined with *P* and the bisecting line is drawn, whose segment between *P* and the *X* axis we call e_x. Next, e_x is translated horizontally until its bottom end coincides with the origin of coordinates *O* (the translated segment $\overline{OP_1}$ has also been labeled e_x in the figure, to indicate its length). This, like many other elementary constructs, is trivially achievable with straightedge and compass. We call P_1 its upper end and *r* the straight line by *O* and P_1 (we only draw part of it in the first quadrant). With center at *O*, P_1 is rotated onto axis Y^+, calling P_2 the point obtained.

Let us call P_{aX}, P_{bX}, P_{aY}, P_{bY} the points of coordinates $(a,0)$, $(b,0)$, $(0,a)$, $(0,b)$, respectively. By P_2 a parallel to $\overline{P_{bY}P_1}$ is drawn until intersecting *r* at P_3, which is in turn rotated upon *Y*, calling P_4 the

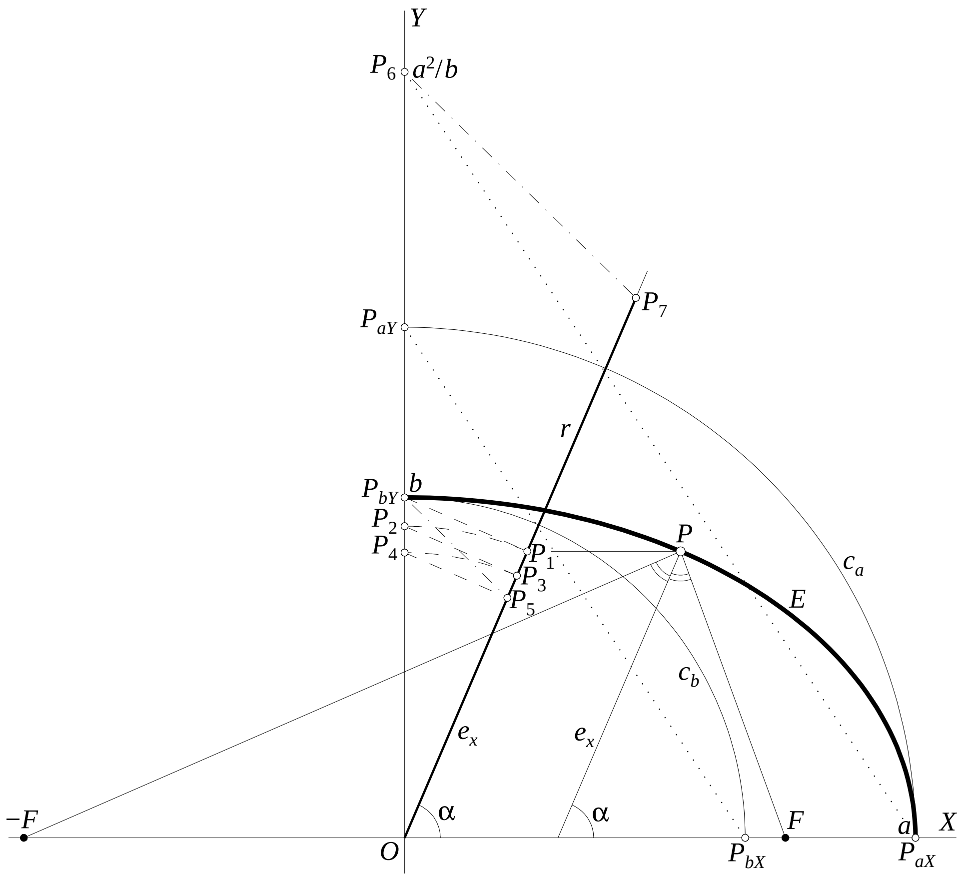

Fig.1. Radius of curvature of an ellipse with straightedge and compass: First Construction.

point obtained. Again the parallel to $\overline{P_{bY}P_1}$ by P_4 is drawn to meet r at P_5.

Besides, the parallel to $\overline{P_{bX}P_{aY}}$ by P_{aX} is drawn to meet Y^+ at P_6. And finally, the parallel to $\overline{P_{bY}P_5}$ is drawn by P_6 until intersecting r at P_7. The radius of curvature of E at P is exactly distance $\overline{OP_7}$.

2.1.2 Mathematical Justification of the First Construction

It is well-known that the two bisecting lines of the straight lines joining P with both foci of E are the tangent and the normal to E by P (only the normal e_x is drawn in Fig.1; its perpendicular by P, not drawn is tangent to E). Therefore, by virtue of the above-stated property taken from [2], the length of e_x is proportional to the cube root the radius of curvature R of E at P. More precisely: the ratio between e_x and the cube root of R (searched) is equal to the ratio between half-axis b and the cube root of the radius of curvature at P_{bY}. This is well-known to be a^2/b, so it can be easily obtained, by the Theorem of Thales, by means of the two dotted parallels in Fig.1 (i.e, the parallel to $\overline{P_{bX}P_{aY}}$ by P_{aX} so as to obtain P_6, of ordinate a^2/b). Therefore, to solve the problem, it is enough to raise the ratio e_x/b to the third power (which is achieved by means of the two dashed rotations upon Y^+ –from P_1 and P_2 to P_4 and P_3, respectively– followed by the corresponding dashed parallels until r –$\overline{P_2P_3}$ and $\overline{P_4P_5}$, which are drawn parallel to $\overline{P_{bY}P_1}$–) and then to multiply the resulting ratio by the radius of curvature at P_{bY} (which is achieved by the two dash-dotted parallels in Fig.1, i.e., $\overline{P_{bY}P_5}$ and $\overline{P_6P_7}$).

2.2 Construction for the Hyperbolic Case

Fig.2 shows the construction to solve the problem for the case of a hyperbola. As will be apparent, the idea behind it is very similar to the previous one. It has been drawn for $a=4$, $b=3$. Point P, of coordinates $(a\cosh t, a\sinh t)$, has been drawn for $t=0.48$.

2.2.1 Description of the Constructive Steps

Again, the positions of foci F and $-F$ of H are determined in the first place (which has not been represented, but is trivial since $F^2 = a^2 + b^2$). Both foci are joined with P and the bisecting line is drawn, the length of whose segment between P and the OY axis at P_1 we call e_y. Using this distance, point P_2 is drawn on the Y axis with ordinate e_y. With center at O, P_2 is rotated onto axis X^+, calling P_3 the point obtained.

By P_3 a parallel to $\overline{P_2a}$ is drawn until intersecting the OY axis at P_4, which is in turn rotated upon X^+, calling P_5 the point obtained. A second parallel by P_5 is drawn to meet the OY axis at P_6.

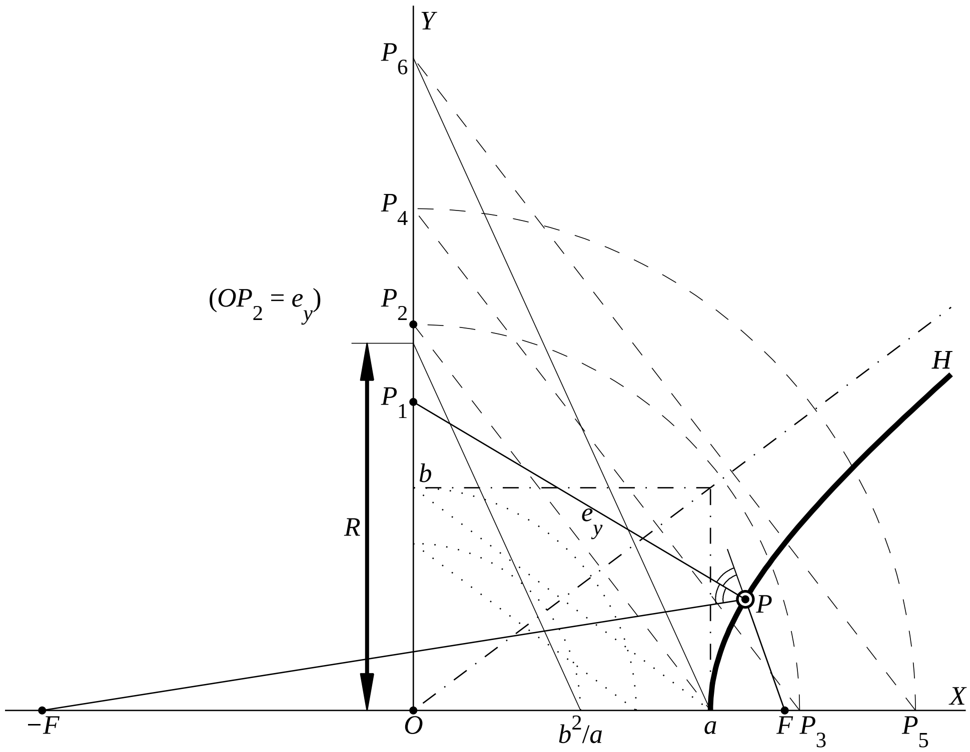

Fig.2. Radius of curvature of a hyperbola at any point with straightedge and compass.

Besides, b is rotated onto the OX axis, from where a parallel to $\overline{ab}$ is drawn to meet the OY axis. When this last point is rotated again onto the OX axis, its abscissa is b^2/a. Drawing from this point a parallel to previously obtained $\overline{P_6 a}$ meets the OY axis at a point whose ordinate is exactly the searched radius of curvature R of the hyperbola at its point P.

2.2.2 Mathematical Justification of the Construction for the Hyperbolic Case

It is well-known that the two bisecting lines of the straight lines joining P with both foci of H are the tangent and the normal to H by P; therefore, by virtue of the property stated in Section 1, the length e_y is proportional to the cube root the radius of curvature R of H at P. More precisely: the ratio between e_y and the cube root of R (searched) is equal to the ratio between half-axis a and the cube root of the radius of curvature at $(a,0)$. This is well-known to be b^2/a, so it can be easily obtained, by the Theorem of Thales, by means of the dotted lines in Fig.2. Therefore, to solve the problem, it is enough to raise the ratio e_y / a to the third power (which is achieved by means of the two dashed rotations upon X^+ followed by the corresponding dashed parallels until OY) and then to multiply the resulting ratio by the already obtained radius of curvature at $(a,0)$.

A very similar construction based on the same principles may be derived for the parabolic case. However, we think it is more interesting to show how the first construction presented, for example, may be optimized in the two senses explained below.

2.3 Second, Optimized Construction for the Elliptical Case

Fig.3 shows the optimized construction proposed.

The main advantage of the first constructions is a pedagogic one, and it lies in the easiness with which it can be reproduced by simple reasoning just knowing that the cube root of the radius of curvature at P is proportional to the length of normal until any axis of the conic.

This last construction is a variation that requires less paper area and less precision in the determination of intersections. It also results in a more harmonic (uniform) distribution of lines on the drawing.

It requires less paper area because the points used are nearer origin of coordinates O, being all in the same quadrant as P (first one, in our case). Besides, it is not necessary to draw the largest radius of curvature of E, which is at P_{bY} if $a>b$.

To illustrate this difference, it may be noted that Fig.1 was drawn with $a = 1.5$, $b = 1$ (and $t = 1\,\text{rad}$), while Fig.3 is drawn with $a = 2$, $b = 1$ (and equally $t = 1\,\text{rad}$), so the largest radius of curvature has effectively changed from $a^2/b = 1.5^2/1 = 2.25$ to $2^2/1 = 4$. This means that Fig.1, if drawn with the data of Fig.3, would have needed a considerable reduction to fit in the page, while Fig.3 can be seen to fit quite comfortably even in just one text column.

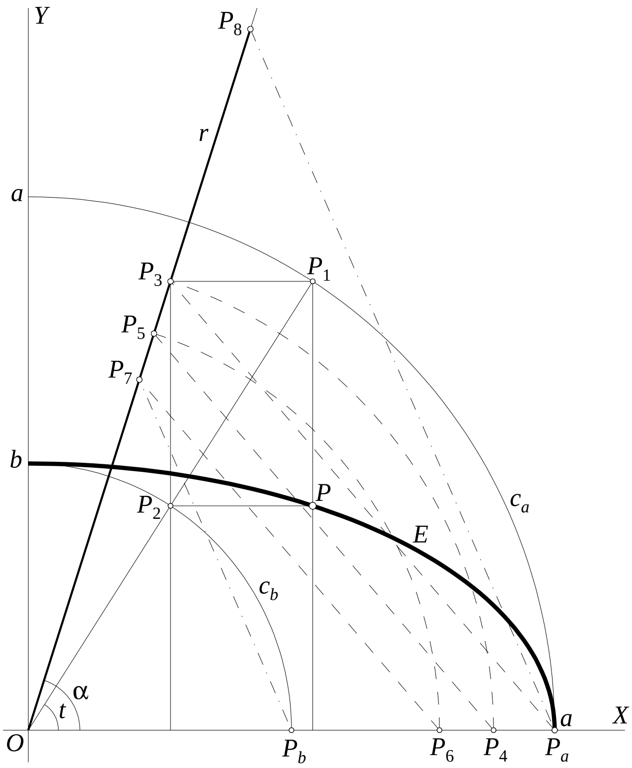

Fig.3. Radius of curvature of an ellipse with straightedge and compass: Second Construction.

This second elliptical construction also requires less precision in the determination of intersections because none takes place under small angles, and all the segments to which parallels have to be drawn are quite longer than $\overline{P_{bY}P_1}$ and, especially, $\overline{P_{bY}P_5}$ in the First Construction. In other words, the conditioning of the problem is better with this method.

2.3.1 Description of the Constructive Steps

As with the previous ones, we will first describe the steps to carry out the construction, and then we will go on to justify its mathematical exactitude. The data are the same as before: half-axes a, b of ellipse E, and its point P in which its radius of curvature must be calculated. Again, the ellipse itself is also represented for illustrative reasons, even if only a discrete number of its points may be drawn by the sole use of straightedge and compass.

We start by drawing the main circle c_a and the minor circle c_b of E (only their portions in the first quadrant are drawn). We call P_a and P_b their respective intersecting points with X^+. By P a vertical line is drawn until c_a at P_1, and a horizontal until c_b at P_2; horizontal by P_1 and vertical by P_2 meet at P_3, and we call r the straight line by coordinate origin O and P_3.

With center at O, P_3 is rotated onto X^+ at P_4; by P_4 parallel to $\overline{P_aP_3}$ until r at P_5, which is again rotated onto P_6, and by P_6 another parallel to $\overline{P_aP_3}$ until P_7. Finally, by P_a a parallel to $\overline{P_bP_7}$ is drawn until r at P_8, being distance $\overline{OP_8}$ the exact radius of curvature of E at P.

2.3.2 Mathematical Justification of the Second Construction

The coordinates of point P of the ellipse are, as can be seen in Fig.2, $(a\cos t, b\sin t)$. Note that parameter t is not the angle defined by the OX axis and the position vector of P itself, but of auxiliary points P_1 and P_2, which are aligned with the origin of coordinates O.

By the Theorem of Pythagoras, $\overline{OP_3}^{\,3}$ is the numerator of well-known equation:

$$R = \frac{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}{ab} \tag{1}$$

as e.g. in [2], p. 796, so it will be enough to raise $\overline{OP_3}$ to the cube and divide by $a\,b$ to solve the problem. Assuming, to simplify the explanation, $a = 1$, and by the Theorem of Thales, both rotations upon X^+ followed by parallels until r do raise $\overline{OP_3}$ to the third power, while the dash-dotted parallels achieve the division by $a\,b = b$. Finally note that assuming $a = 1$ implies no loss of generality, because the unit of length can always be chosen to be equal to the length of segment $\overline{OP_a}$ (i.e., a), and if the radius of the osculating circle is the $\overline{OP_8}$ segment drawn, this will hold whatever length unit is used.

Finally, note that due to the also well-known equations

$$\begin{cases} \cos\alpha = b\cos t \Big/ \left(a^2\sin^2 t + b^2\cos^2 t\right)^{1/2} \\ \sin\alpha = a\sin t \Big/ \left(a^2\sin^2 t + b^2\cos^2 t\right)^{1/2} \end{cases} \tag{2}$$

as e.g. in [2], p. 796, the angle shown as α on Fig.2 is indeed the one of the normal to E at P, which would save the representation of $-F$, perhaps too far on the left, in the First Construction as well.

3 Conclusions and Further Work

New geometric constructions to find the radius of curvature of conics at any one of their points by means of straightedge and compass have been presented. These constructions are consequences of a newly discovered property of conic sections with many engineering and constructive applications [2].

Deeper comparison with already existing methods [5], [6], and a study from the projective viewpoint [7] may constitute the object of further investigation.

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