Optimal Receding Horizon Filter for Continuous-Time Nonlinear Stochastic Systems

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Abstract: A receding horizon filtering problem for nonlinear continuous-time stochastic systems is considered. The paper presents the optimal receding horizon filtering equations. Derivation of the equations is based on the Kushner-Stratonovich and Fokker-Planck-Kolmogorov equations for conditional and unconditional density functions. This result could be a theoretical basis for the optimal control in nonlinear stochastic systems with incomplete information over the most recent time interval. The approximate solutions of the optimal receding horizon filtering equations are discussed. In particular, for linear stochastic systems, the optimal linear receding horizon filter represents the combination of the Kalman and Lyapunov equations. Simulation result is provided.

Key-Words: receding horizon filter, Kushner-Stratonovich equation, Kalman filter, conditional density

1 Introduction

In the literature, the problem of estimating the state of a dynamic systems by using only the information over the most recent time interval is defined in various ways, e.g., finite-memory, receding-horizon or sliding-window estimation. This problem was originally faced in [1]. Later on, the importance of such estimation methods has grown in many application areas where model uncertainties prevent one from successfully using linear Kalman filtering techniques [2]-[8] (see also the references cited therein).

The design of receding horizon filters for general nonlinear systems is still a difficult problem. This is because of the high level concepts associated to the classical nonlinear approaches. A general filtering problem may be stated in a deterministic framework or in a stochastic one. In [8], the design and convergence of receding horizon nonlinear observers for deterministic systems are given.

In this paper, we address the receding horizon filtering problem within nonlinear continuous-time stochastic systems. The main goal of this paper is to derive optimal receding horizon filtering equations. We also demonstrate that these equations are the basis for designing of the optimal linear and suboptimal nonlinear receding horizon filters.

This paper is organized as follows. The problem of receding horizon filtering for nonlinear dynamic systems described by Ito stochastic differential equations is stated in Section 2. In Section 3, we derive an optimal receding horizon filtering equations. The derivation is based the Kushner-Stratonovich and Fokker-Planck-Kolmogorov equations for the conditional and unconditional density functions. In Section 4, two groups of a suboptimal nonlinear receding horizon filtering algorithms are discussed. The first group is based on a parametrization of density functions by means of orthogonal series expansions. And the second one is using the Taylor approximation of nonlinearities. In Section 5, the optimal linear receding horizon filter is derived. We show that in order to solve the Kalman filter equations, one needs to solve the Lyapunov equations for unconditional mean and covariance. In Section 6, the linear receding horizon filter is numerically tested. Example demonstrates the high-accuracy and robustness of the proposed filter. Finally, Section 7 is the conclusion.

2 Statement of Receding Horizon Filtering Problem

The nonlinear filtering problem considered here is based on the following signal observation model:

$$dx_t = f(x_t, t)dt + g(x_t, t)dv_t, \quad t \ge 0,$$
$$dy_t = h(x_t, t)dt + dw_t, \quad y_0 = 0, \tag{1}$$

where x_t, v_t, y_t, and w_t are random processes with values in $\Re^n, \Re^p, \Re^m$, and $\Re^m$, respectively, and v_t and w_t are independent Wiener processes with $E(dv_t dv_t^T) = Q_t dt$ and $E(dw_t dw_t^T) = R_t dt$. The equations (1) are understood in the Ito sense.

We assume that the initial condition x_0, and v_t, w_t are independent. A priori distribution $p(x_0)$ is assumed given. We refer to x_t as the state of the system at time t and y_t as the observation at time t.

Given the history of the process

$$y_{t-\Delta}^t = \{y_s : \ t-\Delta \le s \le t\} \tag{2}$$

on the horizon $[t-\Delta, t]$, $\Delta > 0$, we wish to find the best mean square estimate of the state x_t.

3 Optimal Nonlinear Receding Horizon Filter

3.1 Evolution of the Conditional Density

The best mean square estimate is given by the conditional expectation

$$\hat{x}_t = E(x_t | y_{t-\Delta}^t) = \int_{\Re^n} x p(x, t | y_{t-\Delta}^t) dx, \tag{3}$$

where $p(x, t | y_{t-\Delta}^t)$ represents the conditional density of the state x_t given the observations $y_{t-\Delta}^t$. It is well known that $p_s \equiv p(x, s | y_{t-\Delta}^s)$ satisfies the Kushner-Stratonovich (KS) equation (see, for example, [1],[8]). We have

$$dp_s = L(p_s)dt + (h_s - \hat{h}_s)^T R_s^{-1}(dy_s - \hat{h}_s ds)p_s, \tag{4}$$
$$t-\Delta \le s \le t,$$

where

$$\hat{h}_s = E[h(x_s, s) | y_{t-\Delta}^s] = \int_{\Re^n} h(x, s) p_s dx,$$

$$L(p_s) = -\sum_{i=1}^n \frac{\partial(p_s f_i(x, s))}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2(p_s \sigma_{ij}(x, s))}{\partial x_i \partial x_j}, \tag{5}$$

$$\sigma(x, s) = g(x, s) Q_s g(x, s)^T, \quad \sigma(x, s) = [\sigma_{ij}(x, s)],$$
$$f(x, s) = [f_1(x, s) \ \ldots \ f_n(x, s)]^T.$$

The initial condition (density) at time $s = t - \Delta$ for equation (6) represents the unconditional density of the state x at time $t - \Delta$, i.e.,

$$p_{t-\Delta} = \tilde{p}(x, t-\Delta). \tag{6}$$

3.2 Evolution of the Initial Density

The initial density $\tilde{p}_\tau = \tilde{p}(x, \tau)$ can be determined by the Fokker-Planck-Kolmogorov (FPK) equation [1], [8]. We get

$$d\tilde{p}_\tau = L(\tilde{p}_\tau)dt, \quad 0 \le \tau \le t-\Delta, \quad \tilde{p}_0 = p(x_0), \tag{7}$$

where $p(x_0)$ is the density function of the initial state x_0. Solution of the FPK equation (7) on the time interval $\tau \in [0, t-\Delta]$ represents the initial density (6) for the KS equation (4).

Thus, the FPK and KS equations completely determine the optimal receding horizon filter (ORHF) for the nonlinear system model (1).

An efficient and exact solution of the optimal receding horizon filtering problems is only possible in some simple cases of the model (1). Naturally, equations (4) and (7) give an exact solution of the optimal receding horizon filtering problem in the general case for any nonlinear equations (1). But as well as in the standard nonlinear filtering this solution is not efficient because it cannot be realized in practice [1], [8].

To find the optimal receding horizon estimate of the state vector (3) it is necessary at first to solve the FPK equation (7) for the initial density (6) and then to solve the KS equation (4) for the conditional density of the state x_t. The FPK equation does not contain the observation data (2) therefore it can be pre-computed. However the KS equation (4) depends on the observation data therefore we may calculate the conditional density and the optimal estimate of the state by formula (3) only after obtaining the results of observations (2).

There exist no methods yet for the exact solution of equations (4) and (7) in the general case. The numerical solution of these equations is also impossible in practical problems, as it is very time consuming. Moreover the FPK and KS equations represent complex partial integro-differential equations, especially for the high-dimension models (1). Thus, we are forced to consider approximate methods for solution of the nonlinear receding horizon filtering problem.

4 Suboptimal Receding Horizon Filtering

4.1 Parameterization of density functions

Some of the approximate methods are based on an approximate solution of the FPK and KS equations by means of the orthogonal series expansions of the conditional p_s and unconditional $\tilde{p}_\tau$ densities, and the others are based on simplification of the exact equations for the optimal state estimate $\hat{x}_s = E\left(x_s \middle| y_{t-\Delta}^s\right)$ and conditional error covariance $P_s = E\left(\tilde{x}_s\tilde{x}_s^T \middle| y_{t-\Delta}^s\right)$, $\tilde{x}_s = x_s - \hat{x}_s$ by means of the Taylor series expansion of the functions f, g and h in the signal model (1). Here we give short survey of the modern methods of approximate solution of the FPK and KS equations.

The simplest and widely used approximate method of solution of the FPK equation (7) is the normal approximation method (NAM) [8]. The generalization of the NAM are the method of moments, method of quasi-moments and other approximate methods based on parametrization of the density function, when the unknown density $\tilde{p}_\tau$ is approximated by a segment of the orthogonal expansion of the density, in particular, a segment of the expansion in Hermite polynomials or a segment of the Edgeworth expansion [1], [8].

As is well-known, the most widely used suboptimal filter is the extended Kalman filter (EKF) for nonlinear filtering problems. The EKF has been successfully applied to numerous nonlinear filtering problems. If nonlinearities are significant, however, its performance can be substantially improved. Such efforts have also been reported in [1], [8].

4.2 Receding Horizon Filter based on EKF and NAM

To design the receding horizon filter we propose to use the EKF and NAM. Then the receding horizon estimate $\hat{x}_s$ and error covariance P_s are determined by the following EKF equations:

$$\begin{aligned} d\hat{x}_s &= f(\hat{x}_s, s)ds + B_s\left[dy_s - h(\hat{x}_s, s)ds\right], \\ dP_s &= \left(F_sP_s + P_sF_s^T - P_sH_s^TR_s^{-1}H_sP_s + \tilde{Q}_s\right)ds, \\ B_s &= P_sH_s^TR_s^{-1}, \quad \tilde{Q}_s = g(\hat{x}_s, s)Q_sg(\hat{x}_s, s)^T, \\ F_s &= \left.\frac{\partial f(x,s)}{\partial x}\right|_{x=\hat{x}_s}, \quad H_s = \left.\frac{\partial h(x,s)}{\partial x}\right|_{x=\hat{x}_s}, \\ & t-\Delta \le s \le t. \end{aligned} \tag{8}$$

We propose to calculate the initial conditions

$$\begin{aligned} \hat{x}_{t-\Delta} &= m_{t-\Delta} \overset{\text{def}}{=} E\left(x_{t-\Delta}\right), \\ P_{t-\Delta} &= K_{t-\Delta} \overset{\text{def}}{=} E\left[x_{t-\Delta}^0\left(x_{t-\Delta}^0\right)^T\right], \\ x_{t-\Delta}^0 &= x_{t-\Delta} - m_{t-\Delta} \end{aligned} \tag{9}$$

for the EKF equations (8) by using the NAM equations. We have

$$\begin{aligned} \dot{m}_\tau &= E_N\left[f(x_\tau, \tau)\right], \quad 0 \le \tau \le t-\Delta, \\ \dot{K}_\tau &= E_N\left[f(x_\tau, \tau)(x_\tau - m_\tau)^T\right] \\ &\quad + E_N\left[(x_\tau - m_\tau)f(x_\tau, \tau)^T\right] \\ &\quad + E_N\left[g(x_\tau, \tau)Q_\tau g(x_\tau, \tau)^T\right], \end{aligned} \tag{10}$$

where the initial conditions $m_0 = E(x_0)$ and $K_0 = E\left((x_0 - m_0)(x_0 - m_0)^T\right)$ are given. And the subscript N at the sign of expectation (E_N) means that it is calculated for the normal distribution $N(m_\tau, K_\tau)$ of the random state vector x_τ.

Thus, the EKF equations (8) for receding horizon estimate and covariance $(\hat{x}_s, P_s)$, and the NAM equations (10) for initial conditions (9) completely establish the suboptimal receding horizon filter (SRHF) for the nonlinear signal model (1).

4.3 Example of SRHF

Consider the scalar signal observation model

$$dx_t = -x_t^3 dt + dv_t, \quad t \geq 0, \\ dy_t = x_t dt + dw_t, \quad y_0 = 0. \tag{11}$$

We assume that $E(d^2 v_t) = qdt$ and $E(d^2 w_t) = rdt$, and $x_0 \sim N(\bar{x}_0, \sigma_0^2)$. Then the SRHF (8)-(10) takes the following form:

$$d\hat{x}_s = -\hat{x}_s^3 ds + \frac{1}{r} P_s (dy_s - \hat{x}_s ds), \\ dP_s = \left(-6\hat{x}_s^2 P_s - \frac{1}{r} P_s^2 + q \right) ds, \\ t - \Delta \leq s \leq t, \quad \hat{x}_{t-\Delta} = m_{t-\Delta}, \quad P_{t-\Delta} = K_{t-\Delta}, \tag{12}$$

where the initial conditions $m_{t-\Delta}$ and $K_{t-\Delta}$ are determined by NAM equations (10),

$$\dot{m}_\tau = -E_N(x_\tau^3) = -3m_\tau K_\tau - m_\tau^3, \\ \dot{K}_\tau = -2E_N[x_\tau^3(x_\tau - m_\tau)] + q \\ = -6K_\tau(m_\tau^2 + K_\tau) + q, \\ 0 \leq \tau \leq t - \Delta, \quad m_0 = \bar{x}_0, \quad K_0 = \sigma_0^2. \tag{13}$$

5 Optimal Linear Receding Horizon Filter

The optimal receding horizon filtering problem may be solved completely in the case of linear equations

$$dx_t = F_t x_t dt + G_t dv_t, \quad t \geq 0, \\ dy_y = H_t x_t dt + dw_t, \quad y_0 = 0, \tag{14}$$

where $F_t \in \Re^{n \times n}$, $G_t \in \Re^{n \times p}$, $H_t \in \Re^{m \times n}$, and initial state x_0 is normal, i.e., $x_0 \sim N(\bar{x}_0, P_0)$. In this case as is well-known, the conditional and unconditional densities are normal, i.e.,

$$p(x, s | y_{t-\Delta}^s) \sim N(\hat{x}_s, P_s), \\ \tilde{p}(x, \tau) \sim N(m_\tau, K_\tau). \tag{15}$$

And the conditional mean $\hat{x}_s$ and covariance P_s are determined by the standard Kalman filter equations

$$d\hat{x}_s = F_s dt + P_s H_s^T R_s^{-1}(dy_s - H_s \hat{x}_s ds), \\ dP_s = \left(F_s P_s + P_s F_s^T - P_s H_s^T R_s^{-1} H_s P_s + \tilde{Q}_s\right) ds, \\ \tilde{Q}_s = G_s Q_s G_s^T, \quad t - \Delta \leq s \leq t, \\ \hat{x}_{t-\Delta} = m_{t-\Delta}, \quad P_{t-\Delta} = K_{t-\Delta}, \tag{16}$$

where the initial conditions $m_{t-\Delta}$ and $K_{t-\Delta}$ are described by the Lyapunov equations

$$\dot{m}_\tau = F_\tau m_\tau, \quad m_0 = \bar{x}_0, \\ \dot{K}_\tau = F_\tau K_\tau + K_\tau F_\tau^T + G_\tau Q_\tau G_\tau^T, \quad K_0 = P_0, \\ 0 \leq \tau \leq t - \Delta. \tag{17}$$

6 Numerical Examples

In this section, two numerical examples are given to illustrate the proposed method. Two examples help us to understand the proposed method compared to the existing method. In the first example, we consider the problem in section 4.3. In the second part, example from section 5 is considered.

6.1 Numerical Example of SRHF

The same model with (11) is considered with uncertainty in the system dynamics. Basically, the performance of EKF is better than SRHF in mean-square error sense. However, the receding horizon strategy gives us advantages when uncertainty is considered in the model [6]. In the example, a system has temporary uncertainty during the interval $20 \leq t \leq 30$.

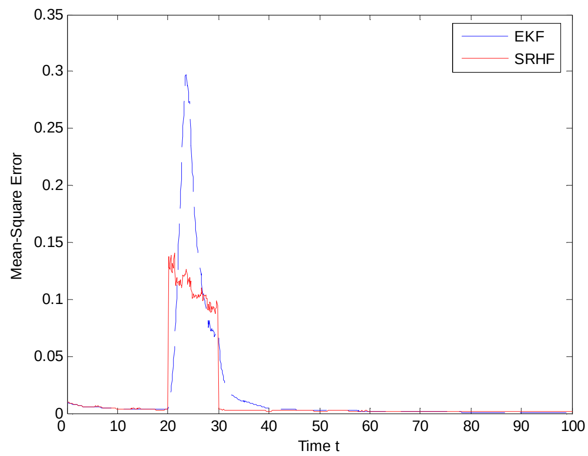

Fig.1 Comparison of MSEs of EKF and SRHF

To compare SRHF with the EKF, Monte-Carlo method is used to calculate mean-square errors. In

the simulation, 1000 experiments are done for Monte-Carlo method. Here, Fig. 1 shows us the MSE comparison of two filters: the EKF and SRHF.
In the figure, during the time interval when the uncertainty occurs, SRHF is superior to the EKF. Because of the limited memory property, SRHF converges to the steady state relatively faster than the EKF when the uncertainty disappears.

6.2 Numerical example of the optimal linear receding horizon filter

The second example considers a linear receding horizon filtering problem in section 5. We take real example from the reference [6].

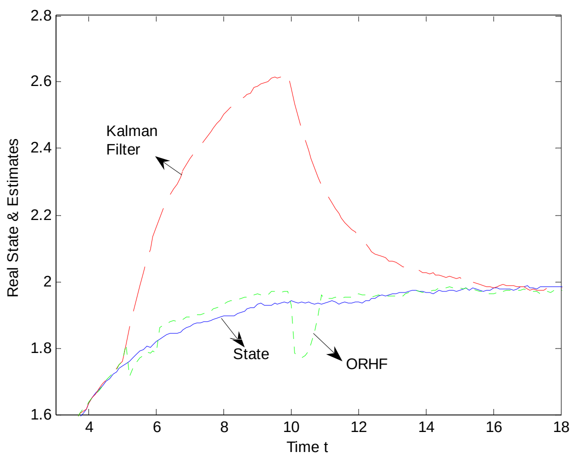

Fig.2 State & two estimates

The Fig.2 gives us quite similar result to that of the previous example. When the temporary uncertainty occurs during the time interval $5 \le t \le 10$, the optimal receding horizon filter (ORHF) gives better estimates and performance. As the uncertainty becomes severer the performance of Kalman filter is getting worse. In other words, ORHF is robust than the Kalman filter against the uncertainty in the model regardless of its degree.

7 Conclusion

The optimal receding horizon filtering equations for nonlinear systems described by Ito stochastic differential equations have been derived. These equations have two-level hierarchical structure. The first level contains the FPK equation for the initial unconditional density, and the second one includes the KS equation for the conditional density. In parallel with the optimal solution of the receding horizon filtering problem we have been proposed the suboptimal solution of this problem based on the EKF and NAM equations. In particular, for linear continuous-time systems we have been derived exact equations for receding horizon filter, which is described by the standard Kalman filtering and Lyapunov equations. Finally, we have been compared the EKF and SRHF for nonlinear case and the Kalman filter and receding horizon filter for linear case in the presence of uncertainty in the dynamics. The simulation example has been demonstrated the high-accuracy and robustness of the proposed filter on the interval of uncertainty.

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