An $L\omega_1\omega_1$ Axiomatization of the Linear Archimedean Continua as Merely Relational Structures

MILOŠ ARSENIJEVIĆ
Department of Philosophy
University of Belgrade
Čika Ljubina 18-20, 11000 Belgrade
SERBIA

MIODRAG KAPETANOVIĆ
Mathematical Institute SANU
Knez Mihailova 35, Belgrade
SERBIA

Abstract. – We have chosen the language $L\omega_1\omega_1$ in which to formulate the axioms of two systems of the linear Archimedean continua – the point-based system, S_P, and the stretch-based system, S_I – for the following reasons: 1. It enables us to formulate all the axioms of each system in one and the same language; 2. It makes it possible to apply, without any modification, Arsenijević's two sets of rules for translating formulas of each of these systems into formulas of the other, in spite of the fact that these rules were originally formulated in a first-order language for systems that are not continuous but dense only; 3. It enables us to speak about an infinite number of elements of a continuous structure by mentioning explicitly only denumerably many of them; 4. In this way we can formulate not only Cantor's coherence condition for linear continuity but also express the large-scale and small-scale variants of the Archimedean axiom without any reference, either explicit or implicit, to a metric; 5. The models of the two axiom systems are structures that need not be relational-operational but only relational, which means that we can speak of the linear geometric continua directly and not only *via* the field of real numbers (numbers will occur as subscripts only, and they will be limited to the natural numbers).

Key-Words: Linear continuum, L_omega_1/omega_1, point-based, stretch-based axiomatization, trivial difference, Archimedean axiom

1 Introduction

Cantor established the *point-based* conception of the continuum, stating that a linearly ordered set of null-dimensional points actually makes up a continuum if the set is *perfect* and *coherent* (*zusammenhängend*) ([7], p. 194). But though the majority of mathematicians and philosophers sided with Cantor's view (cf. [11]), in the last four decades a number of authors revived the Aristotelian *stretch-based* approach (see [1], [3]-[6], [8], [10], [12]-[14], [16]-[19]). However, in spite of the fact that after any axiomatization of each of the two systems – let us call them S_P and S_I, respectively – there will be no model in which the variables of S_P and the variables of S_I range over elements of one and the same basic set, there is a strong intuitive similarity and a possible "systematic connection" between the two systems ([3], p. 84, cf. also [5]) that suggests that they should be classified as only trivially different. The underlying idea is that stretches can be introduced into S_P as intervals between two points while points can be introduced into S_I as abutment places of two stretches (or two equivalence classes of stretches). The fact that stretches are originally neither closed nor open can be compensated by letting them stand for the closed intervals *in contrast* to sets of an infinite number of stretches having either greatest lower or least upper bounds or both, which represent half-open and open intervals, respectively.

2 Problem Formulation

In [2], Arsenijević defined the generalized concepts of trivial syntactical and semantic differences between two formal theories and showed, by using two mutually non-inverse sets of translation rules, that two axiomatic systems implicitly defining point structures and stretch structures that are dense are just trivially different in the defined sense. Now, we want to show that this result holds also when the systems are extended so as to satisfy Cantor's second condition, i.e., if the sructures are not only dense but also continuous. The main problem in showing this consists in the fact that Arsenijević's rules are tailored to first-order languages, whereas the continuity axiom is normally formulated in a second-order language. We shall solve this problem by choosing the language $L\omega_1\omega_1$ in which to formulate the axioms of two systems, which allows the application of Arsenijević's rules without any modification. At the same time, we shall both avoid some unnecessary commitments of the second-order language and always mention only a denumerable number of elements of the continuum.

Another problem is that the two resulting systems of the linear continuum in which numbers are neither mentioned nor used (except as variable subscripts) are insensitive to a distinction between Archimedean and non-Archimedean structures, which both belong to the class of their models (cf. [9]). Since there is no metric, obtainable either geometrically *via* the equality relation holding between stretches or arithmetically through the operations of multiplication and division, the large-scale and the small-scale variant of the Archimedean axiom must be formulated purely topologically by mentioning denumerably many of points and stretches only. This constitutes an important novelty of our approach.

3 Comparison between S_P and S_I

3.1 Axiomatization of the Point-Based System

Let, in the intended model of S_P, the individual variables $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \beta_1, \beta_2, \ldots, \beta_i, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots, \delta_1, \delta_2, \ldots, \delta_i, \ldots, \ldots$ range over a set of null-dimensional points, and let the relation symbols $\equiv$, $<$, and $>$ be interpreted as the identity, precedence, and succession relations respectively. Let the elementary *wffs* of S_P be $\alpha_m \equiv \alpha_n$, $\alpha_m < \alpha_n$, and $\alpha_m > \alpha_n$, where $\alpha_m > \alpha_n \Leftrightarrow_{\text{def.}} \alpha_n < \alpha_m$. Finally, let the axiom schemes of S_P be the following twelve formulas, which we shall refer to as (A_P1), (A_P2),…, (A_P12):

1. $(\alpha_n)\neg\alpha_n < \alpha_n$
2. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l < \alpha_m \wedge \alpha_m < \alpha_n \Rightarrow \alpha_l < \alpha_n)$
3. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \vee \alpha_n < \alpha_m \vee \alpha_m \equiv \alpha_n)$
4. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_l < \alpha_n \Rightarrow \alpha_m < \alpha_n)$
5. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_n < \alpha_l \Rightarrow \alpha_n < \alpha_m)$
6. $(\alpha_m)(\exists\alpha_n)\alpha_m < \alpha_n$
7. $(\alpha_m)(\exists\alpha_n)\alpha_n < \alpha_m$
8. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \Rightarrow (\exists\alpha_l)(\alpha_m < \alpha_l \wedge \alpha_l < \alpha_n))$
9. $(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots((\exists\beta_1)(\wedge_{1\leq i<\omega}\, \alpha_i < \beta_1) \Rightarrow$
   $\Rightarrow (\exists\gamma_1)(\wedge_{1\leq i<\omega}\, \alpha_i < \gamma_1 \wedge$
   $\wedge \neg(\exists\delta_1)(\wedge_{1\leq i<\omega}\, \alpha_i < \delta_1 \wedge \delta_1 < \gamma_1)))$
10. $(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots((\exists\beta_1)(\wedge_{1\leq i<\omega}\, \alpha_i > \beta_1) \Rightarrow$
    $\Rightarrow (\exists\gamma_1)(\wedge_{1\leq i<\omega}\, \alpha_i > \gamma_1 \wedge$
    $\wedge \neg(\exists\delta_1)(\wedge_{1\leq i<\omega}\, \alpha_i > \delta_1 \wedge \delta_1 > \gamma_1)))$
11. $(\exists\alpha_1)(\exists\alpha_2)\ldots(\exists\alpha_n)\ldots(\alpha_2 < \alpha_1 \wedge$
    $\wedge \wedge_{1\leq i<\omega}\alpha_{2i-1} < \alpha_{2i+1} \wedge \wedge_{1\leq i<\omega}\alpha_{2i+2} < \alpha_{2i} \wedge$
    $\wedge (\beta) \wedge_{1\leq i<\omega}(\alpha_i < \beta \wedge \beta < \alpha_{i+2} \Rightarrow$
    $\Rightarrow \wedge_{1\leq k<\omega} \neg\beta\equiv\alpha_k) \wedge (\gamma) \vee_{1\leq i,\, j<\omega}(\alpha_i < \gamma \wedge \gamma < \alpha_j))$
12. $(\exists\alpha_1)\ldots(\exists\alpha_n)\ldots((\beta)\vee_{1\leq i,j<\omega}(\alpha_i < \beta \wedge \beta < \alpha_j) \wedge$
    $\wedge(\gamma)(\delta)(\gamma<\delta \Rightarrow \vee_{1\leq i<\omega} (\gamma<\alpha_k \wedge \alpha_k<\delta)))$

3.2 Axiomatization of the Stretch-Based System

Let, in the intended model of S_I, the individual variables $a_1, a_2, \ldots, a_i, \ldots, b_1, b_2, \ldots, b_i, \ldots, c_1, c_2, \ldots, c_i, \ldots, d_1, d_2, \ldots, d_i, \ldots, \ldots$ range over one-dimensional stretches, and let the relation symbols $=$, $\prec$, $\succ$, $\lmoustache$, $\rmoustache$, $\cap$, and $\Subset$, be interpreted as the identity, precedence, succession, abutment, overlapping, and inclusion relations respectively. Let the elementary *wffs* be $a_m = a_n$, $a_m \prec a_n$, $a_m \succ a_n$, $a_m \lmoustache a_n$, $a_m \rmoustache a_n$, $a_m \cap a_n$, and $a_m \Subset a_n$,

where

$a_m \succ a_n \Leftrightarrow_{\text{def.}} a_n \prec a_m$ and $a_m \rmoustache a_n \Leftrightarrow_{\text{def.}} a_n \lmoustache a_m$,

$a_m \lmoustache a_n \Leftrightarrow_{\text{def.}} a_m \prec a_n \wedge \neg(\exists a_l)(a_m \prec a_l \wedge a_l \prec a_n)$,

$a_m \cap a_n \Leftrightarrow_{\text{def.}} (\exists a_l)(\exists a_k)(a_l \prec a_n \wedge \neg a_l \prec a_m \wedge a_m \prec$
$\prec a_k \wedge \neg a_n \prec a_k)$,

$a_m \Subset a_n \Leftrightarrow_{\text{def.}} \neg a_m = a_n \wedge (a_l)(a_l \cap a_m \Rightarrow a_l \cap a_n)$.

Finally, let axiom schemes of S_I be the following twelve formulas, which we shall refer to as (A_I1), (A_I2),…, (A_I12):

1. $(a_n)\neg a_n \prec a_n$
2. $(a_k)(a_l)(a_m)(a_n)(a_k \prec a_m \wedge a_l \prec a_n \Rightarrow a_k \prec a_n \vee a_l \prec a_m)$
3. $(a_m)(a_n)(a_m \prec a_n \Rightarrow a_m \wr a_n \vee (\exists a_l)(a_m \wr a_l \wedge a_l \wr a_n))$
4. $(a_k)(a_l)(a_m)(a_n)(a_k \wr a_m \wedge a_k \wr a_n \wedge a_l \wr a_m \Rightarrow a_l \wr a_n)$
5. $(a_k)(a_l)(a_m)(a_n)(a_k \wr a_l \wedge a_l \wr a_n \wedge a_k \wr a_m \wedge a_m \wr a_n \Rightarrow$
   $\Rightarrow a_l = a_m)$
6. $(a_m)(\exists a_n)\, a_m \prec a_n$
7. $(a_m)(\exists a_n)\, a_n \prec a_m$
8. $(a_m)(\exists a_n)\, a_n \Subset a_m$
9. $(a_1)(a_2)\ldots(a_i)\ldots((\exists u)(\wedge_{1\le i<\omega}\, a_i \prec u) \Rightarrow$
   $\Rightarrow (\exists v)\, (\wedge_{1\le i<\omega}\, a_i \prec v \wedge \neg(\exists w)\, (\wedge_{1\le i<\omega} a_i \prec w \wedge$
   $\wedge\, w \prec v)))$
10. $(a_1)(a_2)\ldots(a_i)\ldots((\exists u)(\wedge_{1\le i<\omega}\, a_i \succ u) \Rightarrow$
    $\Rightarrow (\exists v)(\wedge_{1\le i<\omega} a_i \succ v \wedge \neg(\exists w)(\wedge_{1\le i<\omega}\, a_i \succ w \wedge$
    $\wedge\, w \succ v)))$
11. $(\exists a_1)(\exists a_2)...(\exists a_n)...$
    $...(a_2 \wr a_1 \wedge \wedge_{1\le i<\omega} a_{2i-1} \wr a_{2i+1} \wedge \wedge_{1\le i<\omega} a_{2i+2} \wr a_{2i} \wedge$
    $\wedge\, (b)\vee_{1\le i,\, j<\omega}\, (a_i \prec b \wedge b \prec a_j))$
12. $(\exists a_1)(\exists a_2)...(\exists a_n)...((b)(\vee_{1\le i<\omega}\, b = a_i \Rightarrow$
    $\Rightarrow\ (\vee_{1\le j<\omega}\, b \wr a_j \wedge \vee_{1\le k<\omega}\, a_k \wr b)) \wedge$
    $\wedge\ (c)(\vee_{1\le i<\omega} c = a_i \Rightarrow \vee_{1\le j<\omega} a_j \Subset c) \wedge$
    $\wedge\, (d)\vee_{1\le i,\, j<\omega}(a_i \prec d \wedge d \prec a_j) \wedge$
    $\wedge\, (e)\vee_{1\le i,\, j<\omega}(a_i \cap e \wedge e \cap a_j))$

3.3 Comments on some Axioms

The interpretation of the first eight axioms of both systems needs no special comments. They implicitly define dense, unbounded, and linearly ordered structures. However, the rest of the axioms need some comments.

Ad (A_P9) and (A_P10), and (A_I9) and (A_I10). - According to Cantor's definition, a linearly ordered set of null-dimensional points is "*perfekt*" (i.e., *dense*) if each element of the set is an accumulation point of an infinite number of elements of the set, whereas it is "*zusammenhängend*" (i.e., *coherent*) if each accumulation point of an infinite number of elements of the set is also an element of the set itself ([7], p. 194). Now, while the first condition is met by axiom (A_P8), the second is met, for the whole class of isomorphic models, only by two axioms, (A_P9) and (A_P10), which state the existence of the least upper and the greatest lower bound, respectively. It might be of interest to note why it is so. Namely, we need both (A_P9) and (A_P10) in order to make the class of all the models for S_P isomorphic. Let us suppose that, though the elements of the intended model of S_P are points, they are, instead (as [in 8], the sets of numbers of closed intervals between any two numbers a and b such that $a \in Q$ and $b \in R$, and $<$ is interpreted as "is a proper subset of". Then, the relational structure $\langle\ \{[a,\ b] \mid a \in Q,\ b \in R\},\ \subset\ \rangle$ satisfies the set of axioms (A_P1),…, (A_P9) but the coherence condition is not met. Let us take, for instance, the set of intervals $[a_1, b]$, $[a_2, b]$,…, $[a_n, b]$,… such that a_1 is a number smaller than b and any a_{n+1} is smaller than a_n, and where π is the accumulation point of the set of numbers a_1, a_2,…, a_n,… . There is no greatest lower bound for this set of intervals, in spite of the fact that the least upper bound always exists. – A similar example can be constructed for showing that we need both (A_I9) and (A_I10).

Ad (A_P11) and (A_I11). The intended meaning of the large-scale variant of the Archimedean axiom can be expressed by choosing a denumerable set of discrete points (in S_P) or abutting stretches (in S_I) distributed over the whole continuum and claiming that for any element of the structure there are two distinct elements (points or stretches) from the given sets such that one of them lies on one side and the other on the other side of the given element (point or stretch). As a consequence, a theorem (whose stretch-based version will be proved below) stating the compactness property of the corresponding structure exhibits the intended meaning of the Archimedean axiom in its most obvious form.

Ad (A_P12) and (A_I12). For precluding infinitesimals in S_P, we have to claim that it is possible to choose a denumerable set of dense points that covers the continuum in such a way that for any two points there is a point from the chosen set that lies between them. In S_I, we have to claim that there are no stretches, like monads in the Robinsonian non-standard field $^*R$ (cf. [15], p. 57), which are impenetrable, from both sides, by some two members of a chosen denumerable set of abutting and dense stretches.

3.4 The Triviality of the Difference between S_P and S_I

In order to show that the two axiom systems, S_P and S_I, are only trivially different in the sense defined in [2], we shall first cite two sets of *translation rules*.

Let f be a function $f : \alpha_n \longrightarrow \langle a_{2n-1}, a_{2n} \rangle$ (n = 1, 2,…) mapping variables of S_P into ordered pairs of variables of S_P, and let C_1-C_5 be the following translation rules providing a 1-1 translation of all the *wffs* of S_P into a subset of the *wffs* of S_I (where $=^C$ means "is to be translated according to syntactic constraints C as"):

C_1: $\alpha_n \equiv \alpha_m =^C a_{2n-1} \wr a_{2n} \wedge a_{2m-1} \wr a_{2m} \wedge a_{2n-1} \wr a_{2m}$,

C_2: $\alpha_n < \alpha_m =^C a_{2n-1} \wr a_n \wedge a_{2m-1} \wr a_{2m} \wedge$
$\wedge a_{2n-1} \prec a_{2m} \wedge \neg a_{2n-1} \wr a_{2m}$,

C_3: $\neg F_P =^C \neg C(F_P)$, where F_P is a *wff* of S_P translated according to C_1-C_5 into *wff* $C(F_P)$ of S_I,

C_4: $F_P' \heartsuit F_P'' =^C C(F_P') \heartsuit C(F_P'')$, where ♥ stands for ⇒ or ∧ or ∨ or ⇔, and F_P' and F_P'' stands for two *wffs* of S_P translated according to C_1-C_5 into two *wffs* of S_I, $C(F_P')$ and $C(F_P'')$ respectively,

C_5: $(\alpha_n)\Omega(\alpha_n) = {}^C(a_{2n-1})(a_{2n})((a_{2n-1} \wr a_{2n}) \Rightarrow$
$\Rightarrow \Omega^*(a_{2n-1}, a_{2n}))$
and
$(\exists\alpha_n)\Omega(\alpha_n) =^C(\exists a_{2n-1})(\exists a_{2n})((a_{2n-1} \wr a_{2n}) \wedge$
$\wedge\, \Omega^*(a_{2n-1}, a_{2n}))$,
where $\Omega(\alpha_n)$ is a formula of S_P translated into formula $\Omega^*(a_{2n-1}, a_{2n})$ of S_P according to C_1-C_5.

Let f^* be a function $f^* : a_n \longrightarrow \langle \alpha_{2n-1}, \alpha_{2n} \rangle$ (n = 1, 2,…) mapping variables of S_I into ordered pairs of variables of S_P, and let C^*_1-C^*_5 be the following translation rules providing a 1-1 translation of all the *wffs* of S_I into a subset of the *wffs* of S_P (where $=^{C^*}$ is to be understood analogously to $=^C$):

C^*_1: $a_n = a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \alpha_{2n-1} \equiv$
$\equiv \alpha_{2m-1} \wedge \alpha_{2n} \equiv \alpha_{2m}$,

C^*_2: $a_n \prec a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge$
$\wedge \neg\alpha_{2m-1} < \alpha_{2n}$,

C^*_3: $\neg F_I =^{C^*} \neg C^*(F_I)$, where F_I is a *wff* of S_I translated according to C^*_1-C^*_5 into *wff* $C(F_I)$ of S_P,

C^*_4: $F_I' \heartsuit F_I'' =^{C^*} C^*(F_I') \heartsuit C^*(F_I'')$, where ♥ stands for ⇒ or ∧ or ∨ or ⇔, and F_I' and F_I'' stands for two *wffs* of S_I translated according to C^*_1-C^*_5 into two *wffs* of S_P, $C^*(F_I')$ and $C^*(F_I'')$ respectively,

C^*_5: $(a_n)\Phi(a_n) =^{C^*}(\alpha_{2n-1})(\alpha_{2n})((\alpha_{2n-1} < \alpha_{2n}) \Rightarrow$
$\Rightarrow \Phi^*(\alpha_{2n-1}, \alpha_{2n}))$
and
$(\exists a_n)\Phi(a_n) =^{C^*}(\exists\alpha_{2n-1})(\exists\alpha_{2n})((\alpha_{2n-1}<\alpha_{2n}) \wedge$
$\wedge\, \Phi^*(\alpha_{2n-1}, \alpha_{2n}))$,
where $\Phi(a_n)$ is a formula of S_I translated into formula $\Phi^*(\alpha_{2n-1}, \alpha_{2n})$ of S_P according to C^*_1-C^*_5.

In [2], Arsenijević has shown that by using C_1-C_5 and C^*_1-C^*_5 for translating (A_P1),…, (A_P8) into S_I and (A_I1),…, (A_I8) into S_P, respectively, we always get theorems. Now, the same holds for the translations of (A_P9),…, (A_P12) into S_I and (A_I9),…, (A_I12) into S_P. Let us prove within S_I the translation of (A_P9), which will be (after an appropriate shortening of the resulting formula) denoted by (A_P9)*.

(A_P9)*

$$(a_1)(a_2)\ldots(a_i)\ldots (\wedge_{1\le i<\omega}\, a_{2i-1} \wr a_{2i} \Rightarrow$$
$$\Rightarrow ((\exists b_1)(\exists b_2)(b_1 \wr b_2 \wedge (\wedge_{1\le i<\omega}\, a_i \prec b_2)) \Rightarrow$$
$$\Rightarrow (\exists c_1)(\exists c_2)(c_1 \wr c_2 \wedge (\wedge_{1\le i<\omega}\, a_i \prec c_2) \wedge$$
$$\wedge\neg(\exists d_1)\neg(\exists d_2)(d_1 \wr d_2 \wedge ((\wedge_{1\le i<\omega}\, a_i \prec d_2) \wedge$$
$$\wedge\, d_1 \prec c_2 \wedge \neg\, d_1 \wr c_2)))).$$

Proof for (A_P9)*

Let us assume both $\wedge_{1\le i<\omega}\, a_{2i-1} \wr a_{2i}$

and

$(\exists b_1)(\exists b_2)(b_1 \mathbin{\{} b_2 \wedge (\wedge_{1\le i<\omega}\, a_i \prec b_2))$,

which are the two antecedents of (A_P9)*. Now, since for any i ($1\le i<\omega$), $a_i \prec b_2$, it follows directly from (A_I9) that there is v such that $a_i \prec v$ and, for no w, both $a_i \prec w$ and $w \prec v$.

Let us now assume, contrary to the statement of the consequent of (A_P9)*, that for any two c_1, c_2 such that $c_1 \mathbin{\{} c_2$ and for any i ($1\le i<\omega$) $a_i \prec c_2$, there are always d_1 and d_2 such that $d_1 \mathbin{\{} d_2$ and for any i ($1\le i<\omega$) $a_i \prec d_2$, so that $d_1 \prec c_2$ and $\neg\, d_1 \mathbin{\{} c_2$. But then, if we take c_2 to be just v from the consequent of (A_I9) (and c_1 any interval such that $c_1 \mathbin{\{} c_2$), the assumption that for any i ($1\le i<\omega$)$a_i \prec c_2$ but $d_1 \prec c_2$ and $\neg\, d_1 \mathbin{\{} c_2$ contradicts the choice of c_2, since if $c_2 = v$, then, according to (A_I9), for any d_1 and d_2 such that $d_1 \mathbin{\{} d_2$ and for any i ($1\le i<\omega$)$a_i \prec d_2$, it cannot be that $d_1 \prec c_2$ and $\neg d_1 \mathbin{\{} c_2$. (Q.E.D.)

4 Application

Let us, finally, prove two theorems in S_I that are of interest for different reasons. The first of them makes clear what is the trick of our formulation of the large-scale version of the Archimedean axiom *via* a chosen denumerable set of abutting stretches distributed over the both sides of the continuum: it is sufficient to have effective control over the continuum by a denumerable number of its discrete elements for making any of its elements surpassable in a finite number of steps, which means that the essence of the Archimedean axiom is topological, having nothing to do with a presupposed metric and depending on no arithmetical operation. The second theorem is a variant of Bolzano-Weierstrass' statement, which turns out to be not only a consequence of the small-scale variant of the Archimedean axiom but also not to be provable without it.

The S_I formulation of the Theorem stating the compactness property for stretches:

$$(c)(d)(c \prec d)\Rightarrow$$
$$\Rightarrow(\exists e_1)(\exists e_2)\ldots(\exists e_m)((e_1 \mathbin{\{} e_2 \wedge \mathbin{\{} \ldots \mathbin{\{} e_m) \wedge$$
$$\wedge (\exists f)(\exists g)(f \mathbin{\{} e_1 \wedge f \prec c \wedge \neg f \mathbin{\{} c \wedge e_{m+1} \mathbin{\{} g \wedge d \prec g \wedge \neg d \mathbin{\{} g))$$

Proof.

Let us choose those i and m, for which a_i and a_{i+m} mentioned in (A_I11) are just those members of the set $a_1, a_2,...,a_n$... for which it holds that $a_i \prec c$ and $d \prec a_{i+m+1}$. Let us take then e_1, $e_2,\ldots e_m$ to be just $a_{i+1}, a_{i+2},\ldots a_{i+m}$. Now, if we take f to be a_i and g to be a_{i+m+1}, we get directly that the statement of the theorem is true.

A stretch-based variant of the Bolzano-Weierstrass Theorem:

$$(c)(d)(h_1)(h_2)\ldots(h_i)...(c \prec d \wedge \neg c \mathbin{\{} d \wedge c \mathbin{\{} h_1 \wedge$$
$$\wedge\ \wedge_{1\le i<\omega}\, h_i \mathbin{\{} h_{i+1} \wedge (\exists e)(e \prec d \wedge \wedge_{1\le i<\omega}\, h_i \prec e)\Rightarrow$$
$$\Rightarrow(\exists b_1)(\exists b_2)\ldots(\exists b_i)\ldots(d \succ b_1 \wedge \wedge_{1\le i<\omega} b_i \mathbin{\}} b_{i+1} \wedge$$
$$\wedge(\exists f)(\exists g)(f \mathbin{\{} g \wedge f \mathbin{\{} (b_i)_{1\le i<\omega} \wedge g \mathbin{\}} (h_i)_{1\le i<\omega})$$

where

$$f \mathbin{\{} (b_i)_{1\le i<\omega} \Leftrightarrow_{\text{def.}} \wedge_{1\le i<\omega} (f \prec b_i) \wedge \neg(\exists v)(f \prec v \wedge$$
$$\wedge \wedge_{1\le i<\omega} (v \prec b_i))$$

and

$$g \mathbin{\}} (h_i)_{1\le i<\omega} \Leftrightarrow_{\text{def.}} \wedge_{1\le i<\omega}(g \succ h_i) \wedge \neg(\exists w)(g \succ w \wedge$$
$$\wedge \wedge_{1\le i<\omega} (w \succ h_i))$$

Proof.

Since the set of stretches $a_1, a_2,\ldots, a_i,\ldots$ from (A_I11) is dense and it holds for each of its members that it abuts some member of the set while some other member abuts it, we can take $b_1, b_2,\ldots, b_i,\ldots$ to be those $a_{j,1}, a_{j,2},\ldots, a_{j,i},\ldots$, respectively, for which the condition $d \succ a_{j1} \wedge \wedge_{1\le i<\omega}\, a_{j,i} \mathbin{\}} a_{j,i+1}$ is met. Now, if f is the greatest lower bound of the set $a_{j,1}, a_{j,2},\ldots, a_{j,i},\ldots$, the statement of the theorem is true. But, let us suppose that, contrary to the statement of the theorem, f is not the greatest lower bound for any set $a_{k1}, a_{k2},\ldots, a_{ki},\ldots$ which is a subset of $a_1, a_2,\ldots, a_i,\ldots$ and which lies within e. This would mean, however, that there is some stretch w that is penetrable by no member of the set $a_1, a_2,\ldots, a_i,\ldots$, which directly contradicts the statement of (A_I12).

5 Conclusion

After formulating in $L\omega_1\omega_1$ the axioms of the Cantorian and the Aristotelian systems of the linear Archimedean continuum, we have shown how, by using appropriate translation rules, the axiom of the point-based system (A_P9), which states the existence of the lowest upper bound, can be proved as a theorem in the stretch-based system. In a similar way, it can be shown that after translating (A_P10), (A_P11), and (A_P12) into S_I, and (A_I9), (A_I10), (A_I11), and (A_I12) into S_P, we also get theorems of S_I and S_P, respectively. This means that S_P and S_I are only trivially different according to Arsenijević's definition given in [2]. In section 4, we have proved, by using the stretch-based system, two important theorems of classical arithmetic. These proofs strongly suggest that other classical theorems concerning the linear Archimedean continuum can also be formulated as being about merely relational structures and proved on the basis of the cited axioms without the use of the algebraic relational-operational structure of real numbers, which presents a prospect for further investigations.

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Acknowledgement: We are grateful to Jerry Massey for advice and support.