The division method in semigroup theory

Rémi Léandre
Institut de Mathématiques de Bourgogne
Université de Bourgogne
B.P. 47870
21078 Dijon Cedex
FRANCE
and
Department of Mathematics
Kyoto University
606-8502 Kyoto
JAPAN

Abstract: Rothschild-Stein division method [21] translated by Léandre in probability [9] is reinterpreted in the framework of the Malliavin Calculus without probability in order to recover the results that Léandre [9,12] has obtained by the Malliavin Calculus.

Key–Words: Hypoelliptic heat-kernels

1 Introduction

The inhomogeneous division method is a very old method for studying subelliptic heat-kernels. For instance Rothschild and Stein used inhomogeneous dilation in order to study the behaviour of heat-kernels associated to degenerated Laplacians on nilpotent Lie groups. Stanton-Tartakoff [24] used this method to study precise estimates on the diagonal for the heat-equation associated to the $\overline{\partial}_b$-Laplacian in complex geometry. Léandre has translated this method in probability in [9]: He considers the asymptotic expansion of a diffusion and introduces some inhomogeneous rescaling in order to study the leading term in the asymptotic expansion of the density of a diffusion, by using the Malliavin Calculus as a tool. Let us remark that Bismut pointed out in [4] that it is not enough to only consider the first fluctuation of a degenerated diffusion for studying the asymptotic in small time of a degenerated diffusion.

We have translated in [15,16,17,18] the Malliavin Calculus of Bismut type in semigroup theory in order to get rough-estimates of some degenerated heat-kernels. Our goal in this paper is to translate in semigroup theory our note [9]. Let us recall the statement: On R^d, we consider some vector fields X_i, $i = 0,..,m$ with bounded derivatives at each order. We suppose that (**Hypothesis H**) the Lie brackets of length smaller than 2 of the X_i's, $i > 0$, span in x_0 the whole space R^d. We consider the operator

$$L = X_0 + 1/2\sum_{i>0} X_i^2 \tag{1}$$

The semigroup spanned by L has got in x_0 a heat-kernel $p_t(x_0, y)$ [19]. For studying precise asymptotic of heat-kernels, it is standard to introduce the bicharacteristics associated to this problem. In our situation, they are the solution starting from (x_0, p_0) of

$$dx_t(x_0, p_0) = \sum_{i>0} < p_t, X_i(x_t(x_0, p_0)) > X_i(x_t(x_0, p_0))dt \tag{2}$$

and of

$$dp_t = -\sum_{i>0} < p_t, X_i(x_t(x_0, p_0)) > {}^t\frac{\partial}{\partial x}X_i(x_t(x_0, p_0))p_t dt \tag{3}$$

The introduction of bicharacteristics for subelliptic estimates is due to Gaveau [3]. We refer to [4] for an introduction to Sub-Riemannian geometry. We can study as it was done for instance by Bismut in [2] the horizontal flow $\Psi_t(x_0, p_0)(z)$ associated to the equation starting from z

$$dz_t = \sum_{i>0} X_i(z_t)h_i(x_0, p_0)(t)dt \tag{4}$$

where $h_i(x_0, p_0)(t) =< p_t, X_i(x_t(x_0, p_0)) >$. $z \to \Psi_s(x_0, p_0)(z)$ is a diffeomorphism on R^d. We set

$$S(x_1(x_0, p_0)) = \sum_{i>0} \int_0^1 h_i^2(x_0, p_0)(t)dt \quad (5)$$

We introduce the following matrix [2]:

$$C = \sum_{i>0} \int_0^1 < \Psi_t^{*-1}(x_0, p_0)X_i(x_0), . >^2 dt \quad (6)$$

We denote by $n(x_0, p_0)$ the dimension of its image plus twice the dimension of its kernel. We remark that this quantity **decreases** when we travel along a small bicharacteristic.

The goal of this note is to provide another proof of the following result by Léandre [9,12] (who was using the Malliavin Calculus) by instead using the Malliavin Calculus of Bismut type without probability of [15].

Theorem 1 *If p_0 is small enough, we have when $t \to 0$*

$$p_t(x_0, x_1(x_0, p_0)) = t^{-n(x_0,p_0)/2}$$
$$\exp[-S(x_1(x_0, p_0))/2t](c(x_1(x_0, p_0)) + o(1)) \quad (7)$$

For developments of the division method by the Malliavin Calculus, we refer to the surveys by Léandre [10,11,13,14], Kusuoka [8] and Watanabe [24], and to the book by Baudoin [1]. For the study of heat-kernels by using probabilistic tools, we refer to the book by Kolokoltsov [6]. For analytical methods for the asymptotics on subelliptic heat-kernels, we refer to [5,7,23].

2 Proof of the theorem

In order to estimate the heat-kernel $p_t(x, y)$ associated to the generator L, we have only to estimate the heat-kernel at time 1 $p_\epsilon(x, y)$ associated to the semigroup P_s^ϵ with generator $\epsilon^2 X_0 + 1/2\epsilon^2 \sum_{i>0} X_i^2$ where we have put $\epsilon^2 = t$. We consider the time dependent generator

$$L^s = \epsilon^2 X_0 +$$
$$1/2\epsilon^2 \sum_{i>0} X_i^2 + \sum_{i>0} h_i(x_0, p_0)(s)X_i \quad (8)$$

We consider the generator $\tilde{L}^s$ on R^{d+1}

$$\epsilon^2 X_0 + 1/2 \sum_{i>0} \tilde{X}_i^2 + \sum_{i>0} h_i(x_o, p_0)(s)X_i \quad (9)$$

where if $i > 0$ $\tilde{X}_i = (\epsilon X_i, -h_i(x_0, p_0)(s))$.

This generator spans a semigroup $\tilde{P}_s^\epsilon$ such that by [16]

$$P_1^\epsilon[f](x_0) = \exp[-S(x_1(x_0, p_0))/2t]$$
$$\tilde{P}_1^\epsilon[\exp[u/\epsilon]f](x_0, 0) \quad (10)$$

We have applied for that the Cameron-Martin-Maruyama-Girsanov formula in semigroup theory of [15]. We use the Itô-Stratonovitch formula of Bismut in semigroup theory of [16]. Let us consider the vector fields

$$\overline{X}_i^s = (\epsilon\Psi(x_0.p_0)_s^{*-1}X_i, -h_i(x_0, p_0)(s)) \quad (11)$$

if $i > 0$ and for $i = 0$

$$\overline{X}_0^s = \Psi(x_0, p_0)_s^{*-1}X_0 \quad (12)$$

and the time dependent generator

$$\overline{L}_s^\epsilon = \epsilon^2 \overline{X}_0^s + 1/2 \sum_{i>0} (\overline{X}_i^s)^2 \quad (13)$$

which spans an inhomogeneous semigroup $\overline{P}_s^\epsilon$. As in [16], in order to estimate $p_t(x_0, x_1(x_0, p_0))$, we have only to estimate the density at 0 of the measure on R^d

$$f \to \overline{P}_1^\epsilon[\exp[u/\epsilon]f(y)](0, 0) \quad (14)$$

The main remark in the sequel is that

$$h_i(x_0, p_0)(s) =< p_0, \Psi(x_0, p_0)_s^{*-1}X_i(x_0) > \quad (15)$$

Therefore if we replace in $\overline{L}_s^\epsilon$ in $\overline{X}_i^s$ for $i > 0$ $h_i(x_0, p_0)(s)$ by $\Psi(x_0, p_0)_s^{*-1}X_i(x_0)$, we consider a semigroup on R^{2d} instead of R^{d+1} and we have to estimate the density at 0 of the measure

$$f \to \overline{P}_1^\epsilon[\exp[< p_0, \epsilon u - y > /\epsilon^2]f(y)](0, 0) \quad (16)$$

where we have kept the same notations for the extended semigroup.

Lemma 2 *There exists a r small enough such that if $|p_0| < r$*

$$\sup_{\epsilon \leq 1} \overline{P}_1^\epsilon[\exp[2 < p_0, \epsilon u - y > /\epsilon^2]](0, 0), \infty \quad (17)$$

Proof: In order to simplify the exposition, we ignore the drift $\overline{X}_0^s$. By proceeding as in [15], we have that

$$|\frac{\partial}{\partial s}\overline{P}_s^\epsilon[\exp[C < p_0, \epsilon u - y >]](0, 0)| \leq$$
$$C^2\epsilon^2\overline{P}_s^\epsilon[\exp[C < p_0, \epsilon u - y >]](0, 0) \quad (18)$$

such that by the Gronwall lemma for $s \leq 1$

$$\overline{P}^{\epsilon}_s[\exp[C < p_0, \epsilon u - y >]](0,0) \leq K \exp[KC^2\epsilon^2 s] \quad (19)$$

such that

$$\overline{P}^{\epsilon}_s[|\epsilon u - y| > M](0,0) \leq K \exp[-KM^2/(\epsilon^2 s)] \quad (20)$$

The same inequality holds for ϵu and y considered separately such that

$$\overline{P}^{\epsilon}_s[| < p_0, u > | > M](0.0) \leq K \exp[-KM^2/s] \quad (21)$$

and

$$\overline{P}^{\epsilon}_s[| < p_0, y > | > M] \leq K \exp[-KM^2/(\epsilon^2 s)] \quad (22)$$

The lemma will be proved if we can show that:

$$\overline{P}^{\epsilon}_s[| < p_0, \epsilon u - y > | > M] \leq K \exp[-KM/\epsilon^2] \quad (23)$$

For that let us introduce as in [17] the Wong-Zakai approximation of the semigroup $\overline{P}^{\epsilon}_s$. Let be a subdivision of $[0,1]$ with smesh η. In order to simplify the exposition, we will suppose that the considered vector fields do not depend on the time s. Let w^i be a centered Gaussian variable on R^m with covariance ηId. Let us introduce the ordinary random differential equation:

$$dx^{\epsilon,\eta}_s(y,u) = \sum_{i>0} \overline{X}_i(x^{\epsilon,\eta}_s(y,u))w^i ds \quad (24)$$

starting from (y,u). We put

$$W.N_{\eta,\epsilon}f(y,u) = E[f(x^{\epsilon,\eta}_1(y,u))] \quad (25)$$

We iterate this kernel k-times and we get a kernel $W.N^k_{\eta,\epsilon}$. An elementary finite-dimensional computation shows that

$$|\Delta W.N^k_{\eta,\epsilon}[\exp[C < p_0, \epsilon u - y >]](0.0)| \leq \eta C^2\epsilon^2 W.N^k_{\eta,\epsilon}[\exp[C < p_0, \epsilon u - y >](0,0) \quad (26)$$

provided that $C\eta^{1/2}\epsilon$ remains bounded. We choose a smesh $\eta = K\epsilon^2 \exp[-K_1 M/\epsilon^2]$. A discrete analogue of the Gronwall lemma shows that $W.N^k_{\eta.\epsilon}$ satisfies to similar inequalities as (19), (20), (21) and (22) uniformly for $k \leq \eta^{-1}$: these 3 last inequalities are obtained by taking $C = C_1 M/\epsilon^2$.

Let C_η be the set of paths from the lattice into R^{2d}. We deduce a probability law $W_{\eta,\epsilon}$ on it. As usual $y^*_{k\eta}$ denotes $\sup_{k' \leq k} |y_{k'\eta}|$.

Moreover, if f is continuous and bounded, we deduce from [16] that

$$W.N^k_{\eta,\epsilon}[f](0,0) \to \overline{P}^{\epsilon}_{k\eta}[f](0,0) \quad (27)$$

uniformly for $k \leq \eta^{-1}$. We regularize the indicatrix function of the cube $[-M,M]^{2d}$. So if the smesh of the subdivision is in $\epsilon^2 \exp[-K_1 M/\epsilon^2]$, it is enough for showing (23) that

$$W.N^k_{\eta,\epsilon}[| < p_0, \epsilon u - y > | > M](0,0) \leq K \exp[-KM/\epsilon^2] \quad (28)$$

if K_1 is small enough uniformly for $k \leq \eta^{-1}$ and in ϵ.

We distinguish two cases: $y^*_{k\eta}$ smaller/greater than $KM^{1/2}$. We consider the function

$$g_k = W_{\eta,\epsilon}[\exp[C < p_0, \epsilon u_{k\eta} - y_{k\eta} >] \,; y^*_{k\eta} < KM^{1/2}] \quad (29)$$

We see that

$$|\Delta g_k| \leq KMC^2\epsilon^2\eta g_k \quad (30)$$

for $M^{1/2}C\epsilon\eta^{1/2}$ bounded. We deduce by a discrete Gronwall lemma that

$$W_{\eta,\epsilon}[\exp[C < p_0, \epsilon u_1 - y_1 >]; y^*_1 \leq kM^{1/2}] \leq K \exp[KMC^2\epsilon^2] \quad (31)$$

provided that $M^{1/2}C\epsilon\eta^{1/2}$ is bounded. Since $\eta = K\epsilon^2 \exp[-k_1 M/\epsilon^2]$, we can take in (31) $C = C_1\epsilon^{-2}$ and deduce that

$$W_{\eta,\epsilon}[| < p_0, \epsilon u_1 - y_1 > | > M; y^*_1 < KM^{1/2}] \leq K \exp[-KM/\epsilon^2] \quad (32)$$

if $M < 1$. If $M \geq 1$, we have the same inequality by the analogue of (20). On the other hand, if K_1 is small enough, we get

$$W_{\eta,\epsilon}[y^*_1 \geq KM^{1/2}] \leq K \exp[-KM/\epsilon^2] \quad (33)$$

In order to show (33), we use the Kolmogorov lemma ([20]). Let X_s be a process $s \in [0,1]$ on a probability space such that $X_0 = 1$ and

$$E[|X_s - X_{s'}|^p]^{1/p} \leq C_p|s - s'|^\alpha \quad (34)$$

for p large enough. Then X_s has a modification which is Hölder and X^*_1 belongs to some L^r with an L^r-norm which can be estimated in terms of the C_p.

We apply this to the measure $W_{\eta,\epsilon}$ and to $X_s = \exp[< C, y_s >]$. Provided that $|C|\epsilon\eta^{1/2}$ is bounded, we find a bound by the considerations which follow (26) of C_p in $\exp[K|C|^2\epsilon^2]$. Therefore we get provided that $|C|\epsilon\eta^{1/2}$ is bounded that

$$W_{\eta,\epsilon}[\exp[< C, y >_1^*]] \leq \exp[KC^2\epsilon^2] \quad (35)$$

(33) arises by taking $|C| = C_1M/\epsilon^2$ ♢

We put $x_0 = 0$. Let $y = y_1 + y_2$ where y_1 is the component of y in the image of the quadratic form C defined in (6) and y_2 its component in the kernel of this quadratic form. We write $y_1 = \Pi_1 y$ and $y_2 = \Pi_2 y_2$.

We do the change of variable

$$y \to 1/\epsilon\Pi_1 y + 1/\epsilon^2\Pi_2 y \quad (36)$$

such that the vector fields $\epsilon\Psi(x_0, p_0)_s^{*-1}X_i$ are transformed as

$$(\Pi_1 + 1/\epsilon\Pi_2)\Psi(x_0, p_0)_s^{*-1}X_i(\epsilon\Pi_1 y + \epsilon^2\Pi_2 y) \quad (37)$$

When $\epsilon \to 0$, these vector fields tend to

$$\Pi_1\Psi(x_0, p_0)_s^{*-1}X_i(0)+ \Pi_2 D\Psi(x_0, p_0)_s^{*-1}X_i(0)\Pi_1 y \quad (38)$$

We consider the generator R_s^ϵ which is got after doing this rescaling and Q_s^ϵ the associated semigroup on functions on R^{2d}.

The main remark is the following: The density at 0 of the measure

$$f \to \overline{P}_1^\epsilon[\exp[< p_0, \epsilon u - y > /\epsilon^2]f(y)](0.0) \quad (39)$$

is equal to $\epsilon^{-n(x_0,p_0)}$ times the density at 0 of the measure

$$f \to Q_1^\epsilon[\exp[< p_0, \epsilon u - y > /\epsilon^2]f(y)](0, 0) \quad (40)$$

We do now the change of variable

$$(y, u) \to (y, \frac{\epsilon u - y}{\epsilon^2}) \quad (41)$$

The vector fields in the extra-components are replaced by

$$- 1/\epsilon\Psi(x_0, p_0)_s^{*-1}X_i(0)+ 1/\epsilon\Psi(x_0, p_0)_s^{*-1}X_i(\epsilon\Pi_1 y + \epsilon^2\Pi_2 y) \quad (42)$$

which tend to $D\Psi(x_0, p_0)_s^{*-1}X_i(0)\Pi_1(y)$ when $\epsilon \to 0$. After performing this rescaling we get an extended generator $\tilde{R}_s^\epsilon$ with a corresponding extended semigroup $\tilde{Q}_s^\epsilon$. We only have to estimate the density at 0 of the measure when $\epsilon \to 0$:

$$f \to \tilde{Q}_1^\epsilon[\exp[< p_0, z >]f(y)](0, 0) \quad (43)$$

We apply for that the Malliavin Calculus without probability of Bismut type of Léandre [16] depending on a parameter.

Lemma 3 *the measure*

$$f \to \tilde{Q}_1^\epsilon[\exp[< p_0, z >]f(y)](0, 0) \quad (44)$$

has a smooth density in y which depends continuously on ϵ belonging to $[0, 1]$.

Proof: We give only the algebraic statement of the proof. It uses the fact (see Lemma 2) that

$$\sup_{0\leq\epsilon\leq 1} \tilde{Q}_1^\epsilon[\exp[2 < p_0, z >]](0.0) < \infty \quad (45)$$

and the Malliavin matrix associated to the generator in the first component with the rescaled vector fields has an inverse bounded in all the L^p, especially for $\epsilon = 0$, since the Hörmander's hypothesis is fulfilled for Lie brackets of length smaller than 2 (**Hypothesis H**). The proof is therefore the same as the one of Theorem III.1 of [16].♢

3 Conclusion

We have translated in semigroup theory the proof of theorem 1 given by Léandre in [8] by using the Malliavin Calculus. The main remark is that in [8] and [9] only algebraic manipulations on stochastic differential equations are performed ans they have their counterparts in semigroup theory. The only notion of the theory of stochastic processes that is used is the elementary Kolmogorov lemma ([20]).

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