New development for expansion of semigroups and application

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Abstract: In this paper we report our research development on expansion of C_0 semigroups. Here we suppose that the generator of the semigroup has discrete spectrum which has separability in some sense. According to the distribution of spectrum, we give various expansion formula. As an application, we give a complete analysis for a controlled string.

Key–Words: expansion of semigroup spectrum Riesz basis controlled string

1 Introduction

Let X be a Banach space and $\mathcal{A}$ be a densely defined and closed linear operator in X. We consider the evolutionary equation in X

$$\begin{cases} \dfrac{d\Phi(t)}{dt} = \mathcal{A}\Phi(t), \quad t > 0, \\ \Phi(0) = \phi_0. \end{cases} \tag{1}$$

If $\mathcal{A}$ generates a C_0 semigroup $T(t)$ on X, Then the solution of (1) can be written as

$$\Phi(t) = T(t)\phi_0. \tag{2}$$

Suppose that spectrum of $\mathcal{A}$ is discrete. Then $\sigma(\mathcal{A}) = \{\lambda_n; n \in \mathbb{N}\}$. Denote by $E(\lambda_n, \mathcal{A})$ the Riesz spectral projector. Recent years, we have devoted to study such a problem that under what conditions one can expand the solution of (1) into

$$\Phi(t) = T(t)\phi_0 = \sum_{n=1}^{\infty} E(\lambda_n, \mathcal{A})T(t)\phi_0 + R(t)\phi_0, \tag{3}$$

where $R(t)\phi_0$ is the residue term.

This assumption is too extensive to study the convergence of the partial sum. In order to obtain more practice assumption for the spectral distribution of $\mathcal{A}$, let us recall an example in control theory.

Consider a controlled Timoshenko beam whose motion is governed by the partial differential equation

$$\begin{cases} \rho\ddot{w}(x,t) - K(w''(x,t) - \varphi'(x,t)) = 0, \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) - K(w'(x,t) - \varphi(x,t)) = 0, \\ w(0,t) = 0, \quad \varphi(0,t) = 0, \\ K(w'(\ell,t) - \varphi(\ell,t)) = -\alpha\dot{w}(\ell,t), \\ EI\varphi'(\ell.t) = -\beta\dot{w}(\ell,t), \end{cases} \tag{4}$$

where I_ρ, ρ, EI, K are physical constants, ℓ is length of the beam , and α and β are positive damping constants.

This model was at first studied by Kim and Renardy [1]. Coleman and Wang in [2], Xu and Feng in [3] analyzed spectrum of this system. They showed that, when $\alpha \neq \rho_1 = \sqrt{\rho/K}, \beta \neq \rho_2 = \sqrt{I_\rho/EI}$, asymptotic spectrum of the system are given by

$$\lambda_{1,n} = \begin{cases} \frac{1}{2\ell}\ln\left|\frac{\alpha-\rho_1}{\alpha+\rho_1}\right| + i\frac{n\pi}{\ell} + o\left(\frac{1}{n}\right), \alpha > \rho_1, \\ \frac{1}{2\ell}\ln\left|\frac{\alpha-\rho_1}{\alpha+\rho_1}\right| + i\frac{(n+1/2)\pi}{\ell} + o\left(\frac{1}{n}\right), \alpha < \rho_1, \end{cases} \tag{5}$$

$$\lambda_{2,n} = \begin{cases} \frac{1}{2\ell}\ln\left|\frac{\beta-\rho_2}{\alpha+\rho_2}\right| + i\frac{n\pi}{\ell} + o\left(\frac{1}{n}\right), \beta > \rho_2, \\ \frac{1}{2\ell}\ln\left|\frac{\beta-\rho_2}{\alpha+\rho_2}\right| + i\frac{(n+1/2)\pi}{\ell} + o\left(\frac{1}{n}\right), \beta < \rho_2, \end{cases} \tag{6}$$

where $n \in \mathbb{N}$. From the above we see that a branch of spectrum diverges to $-\infty$ if $\alpha = \rho_1$ and $\beta \neq \rho_2$, and both branches of spectrum diverge to $-\infty$ if $\alpha = \rho_1$ and $\beta = \rho_2$ hold. There is a similar problem in [4][5].

Based on this practice background, we have pursued our objective in recent years according to the following three types of spectral distribution:

1). Spectrum of $\mathcal{A}$ is of the form $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$, and

$$\sup\{\Re\lambda, \lambda \in \sigma_1(\mathcal{A})\} \le \alpha \le \inf\{\Re\lambda, \lambda \in \sigma_2(\mathcal{A})\}$$

where $\sigma(\mathcal{A}_2) = \{\lambda_n, n \in \mathbb{N}\}$ consists of isolated eigenvalues of $\mathcal{A}$;

2). Spectrum $\sigma(\mathcal{A})$ satisfies condition that for any $a, b \in \mathbb{R}$, set $\sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} \mid |a \le \Re\lambda \le b\}$ is finite;

3). $\sigma(\mathcal{A})$ satisfies the following condition

$$|\Re\lambda_n| \le h, \quad \forall \lambda_n \in \sigma(\mathcal{A}) = \{\lambda_k, k \in \mathbb{N}\}.$$

The first case has more extensive application, the third can be regarded as a special case of the first where $\sigma_1(\mathcal{A}) = \{-\infty\}$. As to the second we need a special treat.

In our study, we always think that one can obtain less information about eigenvectors and root vectors. The main difficulty we encountered is how to obtain much more information from spectral distribution of $\mathcal{A}$. To overcome this difficulty, we employed the exponential function sequence and generalized divided difference of exponential function. About the detail of this trick, we refer readers to our recent contributions[6][7][8][9][11][12].

2 New Research Development

In this section we shall report several new results for expansion of C_0 semigroups on Banach space and Hilbert spaces. Let us begin with two basic notions.

Definition 1 *A sequence $\{\mathcal{H}_j\}_{j=1}^{\infty}$ of subspace of a Hilbert space $\mathcal{H}$ is called a Riesz basis of subspaces (or subspace Riesz basis, see [10]), if any vector $f \in \mathcal{H}$ can be uniquely represented as a series*

$$f = \sum_{j=1}^{\infty} f_j, \quad f_j \in \mathcal{H}_j, \tag{7}$$

and there exist positive constants C_1 and C_2 such that, for each $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty} f_k$,

$$C_1 \sum_{k=1}^{\infty} \|f_k\|^2 \le \|f\|^2 \le C_2 \sum_{k=1}^{\infty} \|f_k\|^2. \tag{8}$$

If subspace sequence $\{\mathcal{H}_j\}_{j=1}^{\infty}$ is a Riesz basis for $\overline{\mathrm{span}\{\mathcal{H}_j; j \ge 1\}}$, then $\{\mathcal{H}_j\}_{j=1}^{\infty}$ is said to be subspace basis sequence, denote by $\mathcal{L}$-basis.

Obviously, if $\dim \mathcal{H}_k \equiv 1, k \in \mathbb{N}$, then there is a sequence $\{e_k, k \in \mathbb{N}\}$ with $\|e_k\| \approx 1$ such that, for any $f \in \mathcal{H}$, there exists uniquely a collection of coefficients $\{c_k(f)\}_{k \in \mathbb{N}}$ such that $f = \sum_{k=1}^{\infty} c_k(f) e_k$ converges in $\mathcal{H}$ and

$$C_1 \sum_{k=1}^{\infty} |c_k(f)|^2 \le \|f\|^2 \le C_2 \sum_{k=1}^{\infty} |c_k(f)|^2.$$

Definition 2 *Let $\Lambda = \{\lambda_n; n \in \mathbb{Z}\}$ be a sequence in $\mathbb{C}$. The sequence Λ is said to be essential space finite separated if there exists a sequence of connected bounded open set, $G^{(p)}, p \in \mathbb{N}$, such that $G^{(p)} \cap G^{(m)} = \emptyset, m \ne p$ and*

$$\Lambda \subset \bigcup_{p \in \mathbb{N}} G^{(p)}, \quad \inf_{p \ne m} \mathrm{dist}(\Lambda^{(p)}, \Lambda^{(m)}) > 0, \tag{9}$$

where $\Lambda^{(p)} = \Lambda \cap G^{(p)}$ and the number of elements including its multiplicity in $\Lambda^{(p)}$ is uniformly bounded.

2.1 The first case

In this subsection we introduce a new result for expansion of semigroup on Hilbert space.

Theorem 3 *Let $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*

1). The spectrum of $\mathcal{A}$ has a decomposition

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A}); \tag{10}$$

2). There exists a real number $\alpha \in \mathbb{R}$ such that

$$\sup_{\lambda \in \sigma_1(\mathcal{A})} \Re\lambda \le \alpha \le \inf_{\lambda \in \sigma_2(\mathcal{A})} \Re\lambda; \tag{11}$$

3). The set $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k \in \mathbb{N}}$ consists of the isolated eigenvalues of $\mathcal{A}$ of finite multiplicity and is essential space finite separated (see, definition 2)

Then the following statements are true:

i). There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$\mathcal{H}_2 = \overline{\mathrm{span}\{\sum_{\lambda \in \sigma_2(\mathcal{A})} E(\lambda, \mathcal{A})\mathcal{H}\}}, \tag{12}$$

and $\mathcal{H}_1 = \{f \in \mathcal{H}, |\ E(\lambda, \mathcal{A})f = 0, \lambda \in \sigma_2(\mathcal{A})\}$, $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2}$ such that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$,

ii). There exists a finite collection, Ω_k, of elements in $\sigma_2(\mathcal{A})$, the corresponding Riesz projector $E(\Omega_k, \mathcal{A})$

$$E(\Omega_k, \mathcal{A}) = \sum_{\lambda \in \Omega_k \cap \sigma_2(\mathcal{A})} E(\lambda, \mathcal{A}), \tag{13}$$

such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}\}_{k\in\mathbb{N}}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$. Therefore, for each $f \in \mathcal{H}_1 + \mathcal{H}_2$, $f = f_1 + f_2$, we have

$$T(t)f = \sum_{k=1}^{\infty} E(\Omega_k, \mathcal{A})T(t)f_2 + T(t)f_1. \tag{14}$$

iii). If $\sup_{k\geq 1} ||E(\lambda_k, \mathcal{A})|| < \infty$, then

$$\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}. \tag{15}$$

iv). $\mathcal{H}$ has the topological direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if and only if

$$\sup_{n\geq 1} \|\sum_{k=1}^{n} E(\Omega_k, \mathcal{A})\| < \infty. \tag{16}$$

When it holds, then for each $f \in \mathcal{H}$, we have $f = f_1 + f_2$, $f_j \in \mathcal{H}_j, j = 1, 2$, and

$$T(t)f = \sum_{k=1}^{\infty} E(\Omega_k, \mathcal{A})T(t)f_2 + T(t)f_1. \tag{17}$$

The following corollary gives a condition such that (16) holds.

Corollary 4 *Let $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$ on $\mathcal{H}$. Suppose that the conditions in Theorem 3 are fulfilled. In addition, if $\{\lambda \in \mathbb{C} \mid \Re\lambda = \alpha\} \subset \rho(\mathcal{A})\}$ and*

$$\sup_{\Re\lambda=\alpha} ||R(\lambda, \mathcal{A})|| < \infty. \tag{18}$$

Then we have decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $T(t) = T_1(t) + T_2(t)$ where $T_j(t) = T(t)|_{\mathcal{H}_j}$. In this situation, $T_2(t)$ is in fact a C_0 group on $\mathcal{H}_2$ and $T_1(t)$ is a C_0 semigroup on $\mathcal{H}_1$.

As a special case of Theorem 3, the following result is evident.

Corollary 5 *Suppose that $\mathcal{A}$ is resolvent compact and the spectrum of $\mathcal{A}$ distributes in a vertical strip $|\Re\lambda| \leq h$ and is essential space finite separated. If the eigenvectors and generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$, Then there is a sequence of generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$ and hence $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{H}$. Therefore for any $f \in \mathcal{H}$, it holds that*

$$T(t)f = \sum_{k=1}^{\infty} E(\Omega_k, \mathcal{A})T(t)f. \tag{19}$$

If $\mathcal{A}$ generates a C_0 group, then the following result is evident.

Corollary 6 *Let $\mathcal{A}$ be the generator of a C_0 group on $\mathcal{H}$. Suppose that $\mathcal{A}$ is resolvent compact and the spectrum of $\mathcal{A}$ is essential space finite separated. Then there is a sequence of generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$ and hence, for any $f \in \mathcal{H}$, it holds that*

$$T(t)f = \sum_{k=1}^{\infty} E(\Omega_k, \mathcal{A})T(t)f. \tag{20}$$

The following result gives an estimation of the residual term $T_1(t)f$ in a non-complete case of the eigenvectors and generalized eigenvectors of $\mathcal{A}$.

Theorem 7 *Let $\mathcal{A}$ generate a C_0 semigroup $T(t)$ and let $R(\lambda, \mathcal{A})f$ is meromorphic function of finite exponential type $\eta(f)$ for each $f \in \mathcal{H}$. Suppose that the spectrum of $\mathcal{A}$ lies in the strip $|\Re\lambda| < h$ and essential space finite separated. Let $\mathcal{H}_j, j = 1, 2$, be defined as in Theorem 3. Then*

1. *for each $f \in \mathcal{H}_1 + \mathcal{H}_2$, we have*

$$T(t)x = \sum_{k=1}^{\infty} E(\Omega_k, \mathcal{A})T(t)f + T_1(t)f, \tag{21}$$

and

$$T_1(t)f \equiv 0 \quad t > \eta(f),$$

where $T_1(t)$ is the restriction of $T(t)$ on $\mathcal{H}_1$.

2. *if*

$$\sup_{\Re\lambda=-h-\varepsilon} \|R(\lambda, \mathcal{A})\| < \infty, \tag{22}$$

is fulfilled, then for any $f \in \mathcal{H}$,

$$T(t)f = T_1(t)f + \sum_{k=1}^{\infty} E(\Omega_k, \mathcal{A})T(t)f, \tag{23}$$

and $T_1(t)f \equiv 0, \quad t > \eta(f)$.

Remark 8 *In this case, the separability of spectrum of $\mathcal{A}$ plays an important role for partial expansion of semigroup.*

2.2 The second case

In this subsection we shall report a result with respect to expansion of semigroup on Banach spaces. We begin with some basic notations.

Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Assume that $\mathcal{A}$ has a discrete spectrum, i.e., $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) =$

$\{\lambda_n; n \in \mathbb{N}\}$, and $\lim_{n\to\infty} \Re\lambda_n = -\infty$. We define the $T(t)-$invariant spectral-subspace of X by

$$Sp(\mathcal{A}) := \overline{\text{span}\left\{\sum_{j=1}^{m} E(\lambda_j, \mathcal{A})x \;\Big|\; x \in X; \forall\, m \in \mathbb{N}\right\}},$$

and another $T(t)-$invariant subspace by

$$\mathcal{M}_\infty := \{x \in X \mid E(\lambda; \mathcal{A})x = 0, \forall \lambda \in \sigma(\mathcal{A})\}.$$

Clearly. $Sp(\mathcal{A}) \cap \mathcal{M}_\infty = \{0\}$, and $\overline{Sp(\mathcal{A}) + \mathcal{M}_\infty} \subseteq X$.

For each $\lambda_n \in \sigma(\mathcal{A})$, denote by m_n the algebraic multiplicity of λ_n, and define operators

$$D_n := (\mathcal{A} - \lambda_n)E(\lambda_n, \mathcal{A}) \text{ and } D_n^0 = E(\lambda_n, \mathcal{A}).$$

Then for each $n \in \mathbb{N}$, D_n is a bounded linear operator with the property that

$$D_n^k = (\mathcal{A} - \lambda_n)^k E(\lambda_n, \mathcal{A}) \quad \text{and} \quad D_n^{m_n} = 0.$$

Now we state main result in Banach space.

Theorem 9 *Let $T(t)$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that $\mathcal{A}$ satisfies the following conditions:*

(c1). there exist positive constants M_1 , ρ_1 and ρ_3 such that

$$\sum_{k=0}^{m_n} \frac{t^k \|D_n^k\|}{k!} \le M_1 e^{-\rho_1 \Re\lambda_n} e^{\rho_3 t}, \quad \forall n \in \mathbb{N}. \tag{24}$$

(c2). there exists a $\tau_0 > 0$ such that the series $\sum_{n=1}^{\infty} e^{\Re\lambda_n \tau_0}$ converges.

Then we can define two family of operators parameterized on $[\tau_0 + \rho_1, \infty)$,

$$T_2(t) : X \to Sp(\mathcal{A}) \quad \textit{and} \quad T_1(t) : X \to \mathcal{M}_\infty,$$

where

$$T_2(t) = \sum_{n=1}^{\infty} E(\lambda_n, \mathcal{A})T(t), \tag{25}$$

such that

1). $T_2(t)$ is a compact operator, $T_1(t)$ and $T_2(t)$ are strongly continuous;

2). $T_j(t)T(s) = T(s)T_j(t) = T_j(t + s)$ for $t \ge \tau_0 + \rho_1,\ s \ge 0,\ j = 1, 2;$

3). $T(t)$ has a decomposition $T(t) = T_1(t) + T_2(t), t \ge \tau_0 + \rho_1$.

In addition, if the following spectral condition holds:

(c3). there exist constants $M_2 > 0$ and $\rho_2 > 0$ such that

$$|\Im\lambda_n| \le M_2 e^{-\rho_2 \Re\lambda_n},$$

then, for each $x \in X$, $T_2(t)x$ is differentiable in $(\tau_0 + \rho_1 + \rho_2, \infty)$.

Remark 10 *In theorem 9, the condition (c1) is a condition on the action of $\mathcal{A}$ on each root subspace. If we take $t = 0$, then we condition (c1) is*

$$\|E(\lambda_n, \mathcal{A})\| \le M_1 e^{-\rho_1 \Re\lambda_n}.$$

Therefore, the condition (c1) includes the case that $\sup_n ||E(\lambda_n, \mathcal{A})|| = \infty$. Also, it requires that $\|E(\lambda_n; \mathcal{A})\|$ grows not faster than $e^{-\rho_1 \Re\lambda_n}$ as $\Re\lambda_n \to -\infty$.

The conditions (c2) and (c3) are requirements on the spectral distribution of $\mathcal{A}$. The condition (c3) is also a spectral condition for the differentiable semigroup (see, [13]). It is equivalent to the condition

$$|\lambda_n| \le M_2 e^{-\rho_2 \Re\lambda_n}, \quad \forall n \in \mathbb{N}.$$

The following result gives an estimate for the residual term.

Corollary 11 *Let $T(t)$ be a C_0 semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that conditions (c1)–(c3) in Theorem 9 hold. In addition, if one of the following conditions is fulfilled:*

1). the generalized eigenvectors of $\mathcal{A}$ are complete in X;

2). the restriction of the resolvent of $\mathcal{A}$ to $\mathcal{M}_\infty$ is an entire function with values in X of finite exponential type h;

then we have

$$T(t) = T_1(t) + T_2(t), \quad t \ge \tau_0 + \rho_1.$$

and for $t > \tau_1$

$$T(t)x = T_2(t)x, \quad t \ge \tau_1, \quad \forall x \in X$$

is a differentiable semigroup for $t > \tau_1$, where

$$\tau_1 := \max\{\tau_0 + \rho_1 + \rho_2, \tau_0 + \rho_1 + h\}, \tag{26}$$

and $T_2(t)$ is given by (25).

3 Application

In this section we shall give an example which comes from control theory.

Let us consider the following controlled string system

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0,\ x \in (0,1), \\ w(0,t) = 0, \\ w(1,t) = -k\int_0^1 x w_t(x,t)dx, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{27}$$

where k is positive feedback gain constant.

We chose state space $\mathcal{H}$ as

$$\mathcal{H} = L^2[0,1] \times H^{-1}[0,1]$$

equipped inner product

$$\begin{aligned} &< (f_1, g_1), (f_2, g_2) >_H := \int_0^1 f_1(x)\overline{f_2(x)}dx \\ &+ \int_0^1 \left[\int_0^x g_1(s)ds - \int_0^1 dr \int_0^r g_1(s)ds\right] \\ &\times \overline{\left[\int_0^x g_2(s)ds - \int_0^1 dr \int_0^r g_2(s)ds\right]}dx. \end{aligned}$$

Clearly, $\mathcal{H}$ is a Hilbert space.

Define an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ (f,g) \in H^1(0,1) \times L^2[0,1] \;\middle|\; \begin{array}{l} f'' \in H^{-1}[0,1], f(0) = 0, \\ f(1) = -k\int_0^1 sg(s)ds \end{array} \right\} \quad (28)$$

$$\mathcal{A}(f,g) = (g, f''), \quad \forall (f,g) \in D(\mathcal{A}). \quad (29)$$

Then the equation (27) can be rewritten as an abstract evolutionary equation

$$\begin{cases} \frac{d}{dt}(w(x,t), w_t(x,t)) = \mathcal{A}(w(x,t), w_t(x,t)), t > 0 \\ (w(x,0), w_t(x,0)) = (w_0(x), w_1(x)), \end{cases} \quad (30)$$

where $(w_0(x), w_1(x)) \in \mathcal{H}$ is given.

A direct verification shows that the following results are true.

Theorem 12 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, then $\mathcal{A}$ is a dissipative operator and has compact resolvent, and hence $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$.*

Theorem 13 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, then we have*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \Delta(\lambda) = 0\}, \quad (31)$$

where

$$\Delta(\lambda) = [(1+k)\lambda - k]e^{\lambda} - [(1-k)\lambda + k]e^{-\lambda}. \quad (32)$$

For each $\lambda \in \sigma(\mathcal{A})$, λ is a simple eigenvalue, corresponding an eigenfunction is given by

$$\Phi_\lambda = (\sinh \lambda x, \lambda \sinh \lambda x). \quad (33)$$

In particular, the set $\{\Phi_\lambda \mid \lambda \in \sigma(\mathcal{A})\}$ is complete in $\mathcal{H}$.

Now we are in a position to determine the asymptotic distribution of $\sigma(\mathcal{A})$.

When $k > 0, k \neq 1$, $\Delta(\lambda)$ defined by (32) has zeros

$$\lambda_n = \begin{cases} \frac{1}{2}\ln\left|\frac{1-k}{1+k}\right| + in\pi + o(\frac{1}{n}), \ 0 < k < 1, \\ \frac{1}{2}\ln\left|\frac{1-k}{1+k}\right| + i\frac{(2n+1)\pi}{2} + o\left(\frac{1}{n}\right), \ k > 1, \end{cases} \quad (34)$$

where $n \in \mathbb{Z}$. It is easy to see from (34) together with Theorem 12 and Theorem 13 that all conditions in Corollary 5 are fulfilled. Therefore we have the following result.

Theorem 14 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, then when $k \neq 1$, the sequence $\{\Phi_{\lambda_n}; n \in \mathbb{Z}\}$ of eigenfunctions of $\mathcal{A}$ forms a Riesz basis for $\mathcal{H}$, and hence $\mathcal{A}$ generates a C_0 group on $\mathcal{H}$.*

If $k > 0, k = 1$, then $\Delta(\lambda)$ becomes

$$\Delta(\lambda) = [2\lambda - 1]e^{\lambda} - e^{-\lambda}. \quad (35)$$

It belongs to the type of equation $a\lambda^m e^{b\lambda} - c = 0$. As shown in [14], it has infinite many zeros, denote them by $\{\lambda_n, n \in \mathbb{N}\}$. Thus

$$|\lambda_n| \leq D_1 e^{-2\Re\lambda_n}, \quad n \in \mathbb{N}, \quad (36)$$

where D_1 is a positive constant.

In what follow, we shall check the conditions in Theorem 9.

Note that $\Delta(\lambda)$ is an entire function of finite exponential type 1 and $\lambda_n, n \in \mathbb{N}$, are zeros of $\Delta(\lambda)$. So we have

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{1+\varepsilon}} < \infty, \quad \forall \varepsilon > 0.$$

Using (36), when $\tau_0 > 2$, we get that

$$\sum_{n=1}^{\infty} e^{\Re\lambda_n\tau_0} \leq D_1 \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{\tau_0/2}} < \infty. \quad (37)$$

So the condition $(c2)$ and $(c3)$ in Theorem 9 are satisfied.

In order to check condition $(c1)$ in Theorem 9, we need to calculate $E(\lambda_n, \mathcal{A})$. Note that

$$D(\mathcal{A}^*) = \left\{ (f,g) \in H^1(0,1) \times L^2[0,1] \;\middle|\; \begin{array}{l} f'' \in H^{-1}[0,1], f(0) = 0 \\ f(1) = k\int_0^1 sg(s)ds \end{array} \right\} \quad (38)$$

$$\mathcal{A}^*(f,g) = -(g, f''), \quad \forall (f,g) \in D(\mathcal{A}^*). \quad (39)$$

For each $\lambda_n \in \sigma(\mathcal{A}), \overline{\lambda}_n \in \sigma(\mathcal{A}^*)$, corresponding an eigenfunction is given by

$$\Psi_{\overline{\lambda}_n} = \eta_n(\sinh \overline{\lambda}_n x, -\overline{\lambda}_n \sinh \overline{\lambda}_n x), \quad (40)$$

where

$$\frac{1}{\eta_n} = -1 + \frac{\cosh 2\lambda_n}{2\lambda_n} - \frac{\sinh 2\lambda_n}{4\lambda_n^3}, \quad (41)$$

and $< \Phi_{\lambda_n}, \Psi_{\overline{\lambda}_n} >_H = 1$ where Φ_{λ_n} is defined as (33). Thus

$$E(\lambda_n, \mathcal{A})F = < F, \Psi_{\lambda_n} >_H \Phi_{\lambda_n}, \quad \forall F \in \mathcal{H}.$$

So we have

$$||E(\lambda_n, \mathcal{A})|| = ||\Psi_{\lambda_n}||||\Phi_{\lambda_n}||.$$

Note that

$$\|(\sinh \lambda x, \lambda \sinh \lambda x)\| = \|(\sinh \overline{\lambda} x, -\overline{\lambda} \sinh \overline{\lambda} x)\|$$

we have

$$||E(\lambda_n, \mathcal{A})|| = |\eta_n|||\Phi_{\lambda_n}||^2.$$

A direct calculation shows that

$$\begin{aligned} ||\Phi_{\lambda_n}||^2 &= \int_0^1 |\sinh \lambda_n x|^2 dx \\ &+ \int_0^1 dx \left| \int_0^x \lambda_n \sinh \lambda_n s ds \right. \\ &\left. - \int_0^1 dr \int_0^r \lambda_n \sinh \lambda_n s ds \right|^2 \\ = & \int_0^1 \left[|\sinh \lambda_n x|^2 + |\cosh \lambda_n x|^2 \right] dx \\ &+ \int_0^1 + \frac{1}{|\lambda_n|^2} |\sinh \lambda_n x|^2 dx \\ &- \frac{1}{|\lambda_n|^2} \int_0^1 [\Re\lambda_n \sinh 2\Re\lambda_n x + \Im\lambda_n \sin 2\Im\lambda_n x] dx \\ = & \frac{\sinh 2\Re\lambda_n}{2\Re\lambda_n} + \frac{1}{2|\lambda_n|^2} \left[\frac{\sinh 2\Re\lambda_n}{2\Re\lambda_n} - \frac{\sin 2\Im\lambda_n}{2\Im\lambda_n} \right] \\ &- \frac{1}{2|\lambda_n|^2} \left[\cosh 2\Re\lambda_n + \cos 2\Im\lambda_n - 2 \right]. \end{aligned}$$

From (36) we can get $|\lambda_n| \simeq e^{-2\Re\lambda_n}$, further $|\sinh 2\lambda_n| \simeq \frac{1}{2}e^{-2\Re\lambda_n}$, $\cosh 2\lambda_n \simeq \frac{1}{2}e^{-2\Re\lambda_n}$. Therefore, we have

$$\begin{aligned} ||E(\lambda_n, \mathcal{A})|| &= |\eta_n|||\Phi_{\lambda_n}||^2 \simeq \frac{|\lambda_n|}{|\Re\lambda_n|} \\ &\leq |\lambda_n| \leq D_1 e^{-2\Re\lambda_n}. \end{aligned}$$

the condition $(c1)$ of Theorem 9 is also fulfilled.

In this situation we have $\rho_1 = \rho_2 = 2, \tau_0 > 2$ and $Sp(\mathcal{A}) = \mathcal{H}$. Applying Theorem 9, we achieve the follow result.

Theorem 15 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, then when $k = 1$ $\mathcal{A}$ generates a differentiable semigroup $T(t)$ on $\mathcal{H}$ for $t > 6$. For $t > 4$ we have*

$$T(t)W_0 = \sum_{n=1}^{\infty} e^{\lambda_n t} < W_0, \Psi_{\lambda_n} >_H \Phi_{\lambda_n}.$$

Acknowledgements: This research is supported by the Natural Science Foundation of China grant NSFC-60474017 and by the Liu Hui Center for Applied Mathematics, Nankai University & Tianjin University.

References:

[1] J. U. Kim and Y. Renady, Boundary Control of the Timoshenko Beam, SIAM. Journal of Control and Optimization, Vol. 25, pp. 1417–1429, 1987

[2] M. P. Coleman and H. WANG, Analysis of the Vibration Spectrum of a Timoshenko Beam with Boundary Damping by the Wave Method, Wave Motion, Vol. 17, pp. 223–239, 1993

[3] G. Q. Xu and D. X. Feng, The Riesz basis property of a Timoshenko beam with boundary feedback and application, IMA Journal of Applied Mathematics, Vol. 67, pp. 357–370, 2002

[4] G. Q. Xu and S. P. Yung, Stabilization of Timoshenko beam by means of pointwise controls. ESAIM Control Optim. Calc. Var. 9 (2003), 579–600 (electronic).

[5] G. Q. Xu, D. X. Feng and S. P. Yung, Riesz basis property of the generalized eigenvector system of a Timoshenko beam, IMA Journal of Mathematical Control and Information (2004) 21, 65–83.

[6] G. Q. Xu, B. Z. Guo, Riesz Basis Property of Evolution Equations in Hilbert Spaces and Application to a Coupled String Equation, SIAM Journal on Control and Optimization. Volume 42, Number 3 pp. 966–984.

[7] G. Q. Xu and D. X. Feng, On the Decomposition of Riesz Operator and the Expansion of Riesz Semigroup Integral Equation and Operator Theory, 2005, 52(4), 581–594.

[8] G. Q. Xu and S. P. Yung, The expansion of semigroup and criterion of Riesz basis, *Journal of Differential equations*, **210**(1) (2005), 1–24.

[9] G. Q. Xu and S. P.Yung, Properties of a class of C_0 semigroup on Banach space and applications, to appear in JMAA.

[10] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, AMS Transl. Math. Monographs **18**(1969).

[11] B. Z. Guo and G. Q. Xu, On basis property of a hyperbolic system with dynamic boundary condition, Differential and Integral Equations, 2005, Volume 18, Number 1, Pages 35-60.

[12] B. Z. Guo and G. Q. Xu, Expansion of solution in terms of generalized eigenfunctions for a hyperbolic system with static boundary condition, Journal of Functional Analysis, 2006, 231(2)245–268

[13] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983, pp.50.

[14] T. Hagen, Asymptotic solution of characteristic equations, *Nonlinear Analysis: Real World Applications*, **6**(2005), 3, 429–446.