A Linear Programming Formulation of the Traveling Salesman Problem

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Abstract: - In this paper, we present a network flow-based, polynomial-sized linear programming formulation of the *Traveling Salesman Problem* (TSP). Computational results are discussed.

Key-Words: - Linear Programming; Network Optimization; Integer Programming; Traveling Salesman Problem; Combinatorial Optimization; Scheduling; Sequencing.

1 Introduction

In this paper, we present a polynomial-sized linear programming formulation of the Traveling Salesman Problem (TSP). The proposed linear program is a network flow-based model. Numerical implementation and results are discussed.

The plan of the paper is as follows. The proposed linear programming formulation is developed in section 2. Numerical implementation and computational results are discussed in section 3. Conclusions are discussed in section 4.

2 Problem Formulation

In this section, we first develop a nonlinear integer programming (NIP) formulation of the TSP. Then, we develop an integer linear programming (ILP) reformulation of this NIP model. Finally, we show that linear programming (LP) relaxation of our ILP reformulation has extreme points that correspond to TSP tours.

2.1 NIP Model

Consider the TSP defined on n nodes belonging to the set N = {1, 2, …, n}, with arc set $E = N^2$, and travel costs t_{ij} ((i,j) $\in$ E; $t_{ii} = \infty$, $\forall$ i$\in$N) associated with the arcs. Assume, without loss of generality, that city 1 is the starting point and the ending point of travel. Denote the set of the remaining cities as M = N \ {1}. Define S = N \ {n} as the index set for the stage of travel corresponding to the order of visit of the cities in M. Let R $\equiv$ S \ {n-1}.

Let u_{is} (i $\in$ M, s $\in$ S) be a 0/1 binary variable that takes on the value "1" if city i $\in$ M is visited at stage s $\in$ S. Re-define the travel costs as:

$$c_{isj} = \begin{cases} t_{ij} + t_{1,i},\ s = 1,\ (i,j) \in M^2; \\ t_{ij},\ s \in R \setminus \{1, n-2\},\ (i,j) \in M^2; \\ t_{ij} + t_{j,1},\ s = n-2,\ (i,j) \in M^2. \end{cases} \quad (2.1)$$

Then, the cost incurred if city i $\in$ M is visited at stage s $\in$ R followed by city j $\in$ M at stage (s+1) can be expressed as $c_{isj}u_{is}u_{j,s+1}$ ((i, j) $\in M^2$, s$\in$R). Note that from expression 2.1 above, $c_{i,1,j}u_{i,1}u_{j,2}$ and $c_{i,n-2,j}u_{i,n-2}u_{j,n-1}$ correctly model the costs of the travels $1 \to i \to j$ and $i \to j \to 1$, respectively.

Hence, the TSP can be formulated as the following nonlinear bipartite matching problem.

Problem TSP:

Minimize

$$ZTSP(u) = \sum_{s\in R}\ \sum_{i\in M}\ \sum_{j\in(M\setminus\{i\})} c_{isj}u_{is}u_{j,s+1} \quad (2.2)$$

Subject to:

$$\sum_{i\in M} u_{is} = 1 \quad s \in S \quad (2.3)$$

$$\sum_{s\in S} u_{is} = 1 \quad i \in M \quad (2.4)$$

$$u_{is} \in \{0, 1\} \quad i \in M;\ s \in S \quad (2.5)$$

□

The objective function 2.2 aims to minimize the total cost of all travels. Constraints 2.3 stipulate (in light of the binary requirements constraints 2.5) that only one city can be visited from city 1 and that only one city is visited at each stage of travel. Constraints 2.4 on the other hand ensure (in light of the binary requirements 2.5) that a given city is visited at exactly one stage of travel. The quadratic objective function terms (i.e., the $c_{isj}u_{is}u_{j,s+1}$'s) ensure (in light of the binary requirements constraints 2.5) that a travel cost is incurred from city i to city j iff those two cities are visited at consecutive stages of travel with i preceding j, as discussed above. Hence, *Problem TSP* accurately models the TSP.

2.2 ILP Model

Note that the polytope associated with *Problem TSP* is the standard assignment polytope (see Bazaraa, Jarvis, and Sherali [1990; pp. 499-513]), and that there is a one-to-one correspondence between TSP tours and extreme points of this polytope. Our modeling consists essentially of *lifting* this polytope in higher dimension in such a way that the quadratic cost function of *Problem TSP* is correctly captured using a linear function. To do this, we use the framework of the graph G = (V, A) illustrated in Figure 2.1, where the nodes in V correspond to (city, travel stage) pairs $(i, s) \in (M, S)$, and the arcs correspond to binary variables $x_{irj} = u_{ir}u_{j,r+1}$ ($(i, j) \in (M, M\setminus\{i\})$; $r \in R$). Clearly, there is a one-to-one correspondence between the perfect bipartite matching solutions of *Problem TSP* (and therefore, TSP tours) and paths in this graph that simultaneously span the set of stages, S, and the set of cities, M. For simplicity of exposition we refer to such paths as "*city and stage spanning*" ("*c.a.s.s.*") paths. Also, we refer to the set of all the nodes of the graph that have a given city index in common as a "level" of the graph, and to the set of all the nodes of the graph that have a given travel stage index in common as a "stage" of the graph.

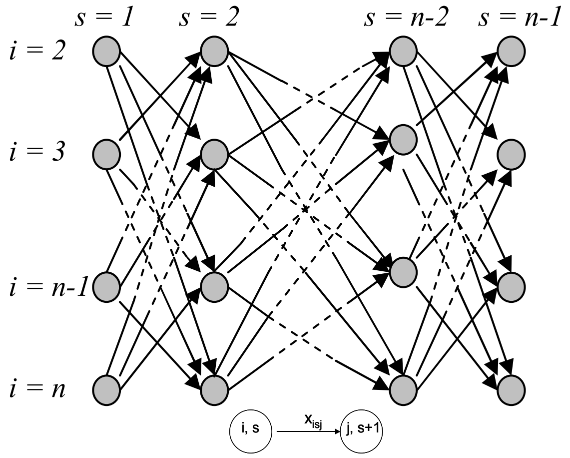

Fig. 2.1: *Illustration of Graph G*

The idea of our approach to reformulating *Problem TSP* is to develop constraints that "force" flow in Graph G to propagate along *c.a.s.s. paths* of the graph only. Hence, we do not deal directly with the TSP polytope *per se* (see Grötschel and Padberg 1985, pp. 256-261]) in this paper. Hence, developments that are concerned with descriptions of the TSP polytope specifically (see Padberg and Grötschel [1985], or Yannakakis [1991] for example) are not applicable in the context of this paper.

For $(i, j, u, v, k, t) \in M^6$, $(p, r, s) \in R^3$ such that $r < p < s$, let $z_{irjupvkst}$ be a 0/1 binary variable that takes on the value "1" if and only if the flow on arc (i, r, j) of Graph G subsequently flows on arcs (u, p, v) and (k, s, t), respectively. Similarly, for $(i, j, k, t) \in M^4$, $(s, r) \in R^2$ such that $r < s$, let y_{irjkst} be a binary variable that indicates whether the flow on arc (i, r, j) subsequently flows on arc (k, s, t) (y_{irjkst} = 1) or not (y_{irjkst} = 0). Finally, denote by y_{irjirj} the binary variable that indicates whether there is flow on arc (i, r, j) or not. Then, with respect to our multi-commodity framework analogy discussed above, we liken y_{irjirj} to a "commodity" that propagates onto stages succeeding stage r in the graph through the y_{irjkst} $(s > r)$ variables. Hence, given an instance of $(\boldsymbol{y}, \boldsymbol{z})$, we use the term "flow layer" to refer to the sub-graph of G induced by the arc (i, r, j) corresponding to a given positive y_{irjirj} and the arcs (k, s, t) $(s \in R,\ s > r)$ corresponding to the corresponding y_{irjkst}'s that are positive. Hence, the flow on arc (i, r, j) also flows on arc (k, s, t) (for a given $s > r$) iff arc (k, s, t) belongs to the *flow layer* originating from arc (i, r, j). Also, we say that flow on a given arc (i, r, j) of Graph G "visits" a given *level* of the graph, say *level* t, if $\sum_{s\in R;\, s\le r-1}\ \sum_{k\in(M\setminus\{i,\, j,\, t\})} y_{tskirj} + \sum_{s\in R;\, s\ge r+1}\ \sum_{k\in(M\setminus\{i,\, j,\, t\})} y_{irjkst} > 0$.

Logical constraints of our model are that: 1) flow must be conserved; 2) flow must be connected; and, 3) *flow layers* must be consistent with one another. By "consistency" of the *flow layers*, we are referring to the requirement that any *flow layer* originating from a given arc (i, r, j) with $r \geq 2$ must be a sub-graph of one or more *flow layers* originating from a set of arcs at any other given *stage* preceding r. More specifically, consider the arc (i, r, j) corresponding to a given positive component of $(\boldsymbol{y})$, $y_{irjirj} > 0$. For $s < r$ $(s \in R)$, define $F_s(i,\ r,\ j) \equiv \{(k, t) \in M^2 \mid y_{kstirj} > 0\}$. Then, by "consistency of flow layers" we are referring to the condition that the *flow layer* originating from arc (i, r, j) must be a sub-graph of the union of the *flow layers* originating from the arcs comprising each of the $F_s(i,\ r,\ j)$'s, respectively. In addition to the logical constraints, the bipartite matching constraints 2.3 and 2.4 of *Problem TSP* must be respectively enforced. These ideas are developed in the following.

1) *Flow Conservations*. Any flow through Graph G must be initiated at *stage 1*. Also, for $(i, j) \in M^2$, $r \in R$, $r \geq 2$, the flow on arc (i, r, j) must be equal to the sum of the flows from *stage 1* that propagate onto arc (i, r, j):

$$\sum_{i \in M} \sum_{j \in M} y_{i,1,j,i,1,j} = 1 \quad (2.6)$$

$$y_{irjirj} - \sum_{u \in M} \sum_{v \in M} y_{u,1,v\,irj} = 0;$$

$i, j \in M;\ r \in R, r \geq 2$ (2.7)

2) *Consistency of "Flow Layers"*. For $p, s \in R$ $(1 < p < s)$ and $(u, v, k, t) \in M^4$, flow on (u, p, v) subsequently flows onto (k, s, t) iff for each $r < p$ $(r \in R)$ there exists $(i, j) \in M^2$ such that flow from (i, r, j) propagates onto (k, s, t) via (u, p, v). This results in the following three types of constraints:

i) *Layering Constraints A*

$$y_{irjupv} - \sum_{k \in M} \sum_{t \in M} z_{irjupvkst} = 0;$$

$i, j, u, v \in M;\ p, r, s \in R, 2 \leq p \leq n\text{-}3,$
$r \leq p\text{-}1,\ s \geq p\text{+}1$ (2.8)

ii) *Layering Constraints B*

$$y_{irjkst} - \sum_{u \in M} \sum_{v \in M} z_{irjupvkst} = 0;$$

$i, j, k, t \in M;\ p, r, s \in R, 2 \leq p \leq n\text{-}3;$
$r \leq p\text{-}1,\ s \geq p\text{+}1$ (2.9)

iii) *Layering Constraints C*

$$y_{upvkst} - \sum_{i \in M} \sum_{j \in M} z_{irjupvkst} = 0;$$

$u, v, k, t \in M;\ p, r, s \in R, 2 \leq p \leq n\text{-}3,$
$r \leq p\text{-}1,\ s \geq p\text{+}1$ (2.10)

3) *Flow Connectivities.* All flows must propagate through the graph, on to stage n-1, in a connected manner. Each *flow layer* must be a connected graph, and must conserve flow:

$$\sum_{k \in M} y_{irjkst} - \sum_{k \in M} y_{irjt,s+1,k} = 0;\quad i, j, t \in M;$$

$r, s \in R, r \leq n\text{-}3, r \leq s \leq n\text{-}3$ (2.11)

$$\sum_{v \in M} z_{vpu\,irjkst} - \sum_{v \in M} z_{u,p+1,v\,irjkst} = 0;$$

$i, j, k, t, u \in M;\ p, r, s \in R,$
$3 \leq r \leq n\text{-}3, s \geq r\text{+}1,\ p \leq r\text{-}2$ (2.12)

$$\sum_{v \in M} z_{irj\,vpu\,kst} - \sum_{v \in M} z_{irj\,u,p+1,v\,kst} = 0;$$

$i, j, k, t, u \in M;\ p, r, s \in R,$
$r \leq n\text{-}5, s \geq r\text{+}3,\ r\text{+}1 \leq p \leq s\text{-}2$ (2.13)

$$\sum_{v \in M} z_{irjkst\,vpu} - \sum_{v \in M} z_{irjkst\,u,p+1,v} = 0;$$

$i, j, k, t, u \in M;\ p, r, s \in R,$
$r \leq n\text{-}5, r\text{+}1 \leq s \leq n\text{-}4,\ s\text{+}1 \leq p \leq n\text{-}3$ (2.14)

4) *"Visit" Requirements*. Flow within any *layer* must *visit* every *level* of Graph G:

$$y_{irjkst} - \sum_{p \in R; p \leq r-1} \sum_{v \in M} z_{upv\,irjkst} + - \sum_{p \in (R \cap [r+1, s-2])} \sum_{v \in M} z_{irj\,vpu\,kst} + - \sum_{p \in R; p \geq s+1} \sum_{v \in M} z_{irjkst\,vpu} = 0;\ r, s \in R, s \geq r\text{+}1;$$

$i, j, k, t \in M;\ u \in M \backslash \{i, j, k, t\}$ (2.15)

5) *"Visit" Restrictions.* Flow must be connected with respect to the stages of Graph G. There can be no flow between nodes belonging to the same *level* of the graph; No *level* of the graph can be *visited* at more than one *stage*, and vice versa:

$$\sum_{(k,t) \in M^2 | (k,t) \neq (i,j)} y_{irjkrt} + \sum_{s \in R; s \geq r+1} \sum_{k \in M} y_{irjksi} + + \sum_{s \in R; s \geq r+1} \sum_{k \in M} y_{irjisk} + \sum_{s \in R; s \geq r} \sum_{k \in M} \sum_{t \in M} y_{irikst} + + \sum_{(k,t) \in (M \backslash \{j\}, M) | (k, r+1, t) \in A} y_{irjk,r+1,t} + + \sum_{s \in R; s \geq r+1} \sum_{k \in M} y_{irjksj} + \sum_{s \in R; s \geq r+2} \sum_{k \in M} y_{irjjsk} + + \sum_{s \in R; s \leq r} \sum_{k \in M} \sum_{t \in M} y_{kstjrj} = 0,$$

$i, j \in M;\ r \in R$ (2.16)

Note that constraints 2.3 of *Problem TSP* are enforced through the combination of the "Flow Connectivities" requirements and the *'Visit' Restrictions* constraints, and that constraints 2.4 are enforced through the *'Visit' Requirements* constraints.

The complete statement of our integer (linear) programming model is as follows:

Problem IP:

Minimize

$$Z_{IP}(\mathbf{y}, \mathbf{z}) = \sum_{r \in R} \sum_{i \in M} \sum_{j \in M} c_{irj} y_{irjirj}$$

Subject to:

Constraints 2.6 – 2.16

$y_{irjkst}, z_{irjupvkst} \in \{0, 1\}\quad i, j, k, t, u, v \in M;$

$p, r, s \in R$

□

The following proposition formally establishes the equivalence between *Problem IP* and *Problem TSP* (The proof is provided in Diaby [2006]).

Proposition 1
Problem IP and *Problem TSP* are equivalent.
□

Hence, each feasible solution to *Problem IP* corresponds to a TSP tour, and conversely. Let $\varphi(\ell) = \langle 1, \ell_1, \cdots, \ell_{n-1}, 1 \rangle$ denote the ordered set of city indices visited along a given TSP tour, Tour ℓ (i.e., with ℓ_t as the index of the city visited at stage t according to Tour ℓ). In the remainder of this paper, we will use the term "feasible solution corresponding to (Given) Tour ℓ" to refer to the vector ($\mathbf{y}(\varphi(\ell))$, $\mathbf{z}(\varphi(\ell))$) obtained as follows:

$$(\mathbf{y}(\varphi(\ell)))_{arbcsd} = \begin{cases} 1 \text{ for } r, s \in R,\ s \geq r, \\ \quad (a,b,c,d) = (\ell_r, \ell_{r+1}, \ell_s, \ell_{s+1}); \\ 0 \text{ otherwise} \end{cases}$$

$$(\mathbf{z}(\varphi(\ell)))_{apbcrdesf} = \begin{cases} 1 \text{ for } p, r, s \in R,\ \ s > r > p, \\ \quad (a, b, c, d, e, f) = \\ \quad = (\ell_p, \ell_{p+1}, \ell_r, \ell_{r+1}, \ell_s, \ell_{s+1}); \\ 0 \text{ otherwise} \end{cases}$$

Our proposed linear programming model will now be developed.

2.3 LP Model

Our basic linear programming model consists of the linear programming relaxation of *Problem IP*. This problem can be stated as follows:

Problem LP:

Minimize

$$Z_{LP}(\mathbf{y}, \mathbf{z}) = \sum_{i \in M} \sum_{r \in R} \sum_{j \in M} c_{irj} y_{irjirj}$$

Subject to:

Constraints 2.6 – 2.16

$$y_{irjkst}, z_{upvirjkst} \in [0,1];\quad u, v, i, j, k, t \in M,$$
$$p, r, s \in R \tag{2.17}$$
□

In the remainder of this section, we establish the equivalence between *Problem LP* and *Problem IP*. (Proofs are given in Diaby [2007]). We begin with the following result.

Lemma 1

The following constraints are valid for *Problem LP*:

i) $y_{irjirj} - \sum_{k \in M} \sum_{t \in M} y_{irjkst} = 0;$

$i, j \in M;\ \ r, s \in R,\ \ s \geq r+1$

ii) $y_{irjirj} - \sum_{k \in M} \sum_{t \in M} \sum_{a \in M} \sum_{c \in M} z_{irjkstabc} = 0;$

$i, j \in M;\ \ r, s, b \in R,\ \ r < s < b$

□

For a feasible solution $(\mathbf{y}, \mathbf{z}) = (y_{irjkst}, z_{upvirjkst})$ to *Problem LP*, let $G(\mathbf{y}, \mathbf{z}) = (V(\mathbf{y}, \mathbf{z}), A(\mathbf{y}, \mathbf{z}))$ be the sub-graph of *G* induced by the arcs of G corresponding to the positive components of ($\mathbf{y}$). For $r \in R$, define $W_r(\mathbf{y}, \mathbf{z}) \equiv \{(i, j) \in M^2 \mid \{(i, r, j) \in A(\mathbf{y}, \mathbf{z})\}$. Denote the arc corresponding to the ν^{th} element of $W_r(\mathbf{y}, \mathbf{z})$ ($\nu \in \{1, 2, \cdots, \chi_r(\mathbf{y}, \mathbf{z})\}$; $1 \leq \chi_r(\mathbf{y}, \mathbf{z}) \leq (n-1)(n-2)$) as $a_{r,\nu}(\mathbf{y}, \mathbf{z})$ = $(i_{r,\nu}, r, j_{r,\nu})$. Then, $W_r(\mathbf{y}, \mathbf{z})$ can be alternatively represented as $X_r(\mathbf{y}, \mathbf{z})$ = $\{(i_{r,\nu}, r, j_{r,\nu});\ \nu \in N_r(\mathbf{y}, \mathbf{z})\}$, where $N_r(\mathbf{y}, \mathbf{z}) = \{1, 2, \cdots, \chi_r(\mathbf{y}, \mathbf{z})\}$ is the index set for the arcs of *Graph* G($\mathbf{y}$, $\mathbf{z}$) originating at *stage* r. For convenience, we will henceforth write $a_{r,\nu}(\mathbf{y}, \mathbf{z})$ simply as $a_{r,\nu}$. Furthermore, we will use a more compact indexing of the $\mathbf{y}$ and $\mathbf{z}$ variables where the set of indices "$i_{r,\nu}, r, j_{r,\nu}$" will be replaced with "($a_{r,\nu}$)", whenever convenient.

For $(r, s) \in R^2$ with $s \geq r+2$, $\rho \in N_r(\boldsymbol{y}, \boldsymbol{z})$, and $\sigma \in N_s(\boldsymbol{y}, \boldsymbol{z})$ we refer to a set of arcs of G($\boldsymbol{y}$, $\boldsymbol{z}$),

$$U_{(r,\rho),(s,\sigma),t}(\boldsymbol{y}, \boldsymbol{z}) \equiv \Big\{ a_{r,\nu_{r,(r,\rho),(s,\sigma),t}}, a_{r+1,\nu_{r+1,(r,\rho),(s,\sigma),t}}, \cdots, a_{s,\nu_{s,(r,\rho),(s,\sigma),t}} \Big| \nu_{r,(r,\rho),(s,\sigma),t} = \rho; \nu_{s,(r,\rho),(s,\sigma),t} = \sigma;$$
$$\nu_{p,(r,\rho),(s,\sigma),t} \in N_p(\boldsymbol{y}, \boldsymbol{z}), \forall p \in (R \cap [r+1, s-1];$$
$$i_{p,\nu_{p,(r,\rho),(s,\sigma),t}} = j_{p-1,\nu_{p-1,(r,\rho),(s,\sigma),t}},\ \forall p \in (R \cap [r+1, s];$$
$$\text{and } z_{(a_{p,\nu_{p,(r,\rho),(s,\sigma),t}}),(a_{q,\nu_{q,(r,\rho),(s,\sigma),t}}),(a_{s,\sigma})} > 0,\ \forall\ (p, q) \in (R \cap [r, s-1])^2 \text{ such that } q > p\ \Big\} \tag{2.18}$$

as a "path in ($\boldsymbol{y}$, $\boldsymbol{z}$) from (r, ρ) to (s, σ)." Hence, for convenience, a *path in (y, z) from (r,ρ) to (s,σ)*, $U_{(r,\rho),(s,\sigma),t}(\boldsymbol{y}, \boldsymbol{z})$, can be alternatively represented as an ordered set of city indices,

$$P_{(r,\rho),(s,\sigma),t}(\boldsymbol{y}, \boldsymbol{z}) = \langle i_{r,\nu_{r,(r,\rho),(s,\sigma),t}}, i_{r+1,\nu_{r+1,(r,\rho),(s,\sigma),t}}, \cdots, i_{s+1,\nu_{s+1,(r,\rho),(s,\sigma),t}} \rangle \tag{2.19}$$

Where:

$\nu_{r,(r,\rho),(s,\sigma),t} = \rho$, $\nu_{s,(r,\rho),(s,\sigma),t} = \sigma$,

$i_{r+1,\nu_{r+1,(r,\rho),(s,\sigma),t}} = j_{r,\rho}$,

$i_{s+1,\nu_{s+1,(r,\rho),(s,\sigma),t}} = j_{s,\sigma}$,

$(i_{p,\nu_{p,(r,\rho),(s,\sigma),t}}, p, i_{p+1,\nu_{p+1,(r,\rho),(s,\sigma),t}}) \in X_p(\boldsymbol{y}, \boldsymbol{z})$,

$\forall\ p \in (R \cap [r, s])$; and

$i_{p,\nu_{p,(r,\rho),(s,\sigma),t}} = j_{p-1,\nu_{p-1,(r,\rho),(s,\sigma),t}},\ \forall p \in (R \cap [r+1, s]$.

Finally, we denote the set of all *paths in (y, z) from (r,ρ) to (s, σ)* as $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z})$, and associate to it the index set $\Psi_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z}) \equiv \{1, 2, \cdots, \varphi_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z})\}$, where $\varphi_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z})$ is the cardinality of $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z})$.

We have the following.

Proposition 2
Let $(\boldsymbol{y}, \boldsymbol{z}) = (y_{irjkst}, z_{upvirjkst})$ be a feasible solution to *Problem LP*. For $(r, s) \in R^2$ $(s \geq r+2)$, $\rho \in N_r(\boldsymbol{y}, \boldsymbol{z})$, and $\sigma \in N_s(\boldsymbol{y}, \boldsymbol{z})$, if $y_{i_{r,\rho}, r, j_{r,\rho}, i_{s,\sigma}, s, j_{s,\sigma}} > 0$, then we must have:

i) $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z}) \neq \varnothing$; and

ii) $\forall$ $g \in (R \cap [r+1, s-1])$ and $\gamma \in N_g(\boldsymbol{y}, \boldsymbol{z})$: $z_{i_{r,\rho}, r, j_{r,\rho}, i_{g,\gamma}, g, j_{g,\gamma}, i_{s,\sigma}, s, j_{s,\sigma}} > 0 \Rightarrow \exists\ \iota \in \Psi_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z}) \ni: (i_{g,\gamma}, j_{g,\gamma}) \in (P_{(r,\rho),(s,\sigma),\iota}(\boldsymbol{y}, \boldsymbol{z}))^2$.

□

Proposition 3
Let $(\boldsymbol{y}, \boldsymbol{z}) = (y_{irjkst}, z_{upvirjkst})$ be a feasible solution to *Problem LP*. Let $(r, s) \in R^2$, $s \geq r+2$; $\rho \in N_r(\boldsymbol{y}, \boldsymbol{z})$; and $\sigma \in N_s(\boldsymbol{y}, \boldsymbol{z})$ be such that $y_{i_{r,\rho}, r, j_{r,\rho}, i_{s,\sigma}, s, j_{s,\sigma}} > 0$. Then, we must have:

i) $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z}) \neq \varnothing$;

Furthermore, for each $\ell \in \Psi_{(r,\rho),(s,\sigma)}(\boldsymbol{y}, \boldsymbol{z})$ we must have:

ii) $i_{q, v_{q,(r,\rho),(s,\sigma),\ell}} = j_{q-1, v_{q-1,(r,\rho),(s,\sigma),\ell}}$
for $q \in R$; $r+1 \leq q \leq s$;

iii) $z_{(a_{p, v_{p,(r,\rho),(s,\sigma),\ell}}),(a_{q, v_{q,(r,\rho),(s,\sigma),\ell}}),(a_{s,\sigma})} > 0$
$\forall$ $(p, q) \in (R \cap [r, s])^2$, $r \leq p < q \leq s-1$;

iv) $i_{p, v_{p,(r,\rho),(s,\sigma),\ell}} \neq i_{q, v_{q,(r,\rho),(s,\sigma),\ell}}$
$\forall$ $(p, q) \in (s \cap [r, s+1])^2 \ni$: $p \neq q$.

□

For convenience, we refer to each $P_{(1,\rho),(n-2,\sigma),k}(\boldsymbol{y}, \boldsymbol{z})$ simply as a "TSP tour in (*y*, *z*)," and denote it by $T_{\rho,\sigma,k}(\boldsymbol{y}, \mathbf{z})$. To a *TSP tour in (y, z)*, $T_{\rho,\sigma,k}(\boldsymbol{y}, \boldsymbol{z})$, we attach a "flow value" $\lambda_{\rho,\sigma,k}(\boldsymbol{y}, \boldsymbol{z})$ defined as:

$$\lambda_{\rho,\sigma,k}(\boldsymbol{y}, \boldsymbol{z}) \equiv \min_{p \in (R \cap [2, n-3])} \left\{ z_{(a_{1,\rho}),(a_{p, v_{p,(1,\rho),(n-2,\sigma),k}}),(a_{n-2,\sigma})} \right\}$$

A set of *TSP tours in (y, z)*, $\Gamma = \{T_{\rho_1,\sigma_1,k_1}(\boldsymbol{y}, \boldsymbol{z}), T_{\rho_2,\sigma_2,k_2}(\boldsymbol{y}, \boldsymbol{z}) \ldots, T_{\rho_m,\sigma_m,k_m}(\boldsymbol{y}, \boldsymbol{z})\}$ with associated set of arc sets in *G*, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ (where $\mathbf{a}_p = \{a_{q, v_{q,(1,\rho_p),(n-2,\sigma_p),k_p}}; q \in [1, n-2]\}$, for $p = 1, \ldots, m$), is said to "cover" (*y*, *z*) if $\bigcup_{1 \leq p \leq m} (\mathbf{a}_p) = A(\boldsymbol{y}, \boldsymbol{z})$. Moreover, we say that (*y*, *z*) "consists of" **Γ** if **Γ** *covers (y, z)* and the following hold:

i) $y_{(a_{r,\rho}),(a_{r,\rho})} = \sum_{p \in [1, m(\mathbf{y},\mathbf{z})] \mid a_{r,\rho} \in \mathbf{a}_p} \lambda_{\rho_p,\sigma_p,k_p}(\mathbf{y}, \mathbf{z})$
$\forall$ $(r, \rho) \in (R, N_r(\mathbf{y}, \mathbf{z}))$;

ii) $y_{(a_{r,\rho}),(a_{s,\sigma})} = \sum_{p \in [1, m(\mathbf{y},\mathbf{z})] \mid (a_{r,\rho}, a_{s,\sigma}) \in (\mathbf{a}_p)^2} \lambda_{\rho_p,\sigma_p,k_p}(\mathbf{y}, \mathbf{z})$
$\forall$ $(r, s) \in R^2$, $(\rho, \sigma) \in (N_s(\mathbf{y}, \mathbf{z}), N_s(\mathbf{y}, \mathbf{z}))$;

iii) $z_{(a_{r,\rho}),(a_{s,\sigma}),(a_{t,\tau})} = \sum_{p \in [1, m(\mathbf{y},\mathbf{z})] \mid (a_{r,\rho}, a_{s,\sigma}, a_{t,\tau}) \in (\mathbf{a}_p)^3} \lambda_{\rho_p,\sigma_p,k_p}(\mathbf{y}, \mathbf{z})$
$\forall$ $(r, s, t) \in R^3$,
$(\rho, \sigma, \tau) \in (N_s(\mathbf{y}, \mathbf{z}), N_s(\mathbf{y}, \mathbf{z}), N_t(\mathbf{y}, \mathbf{z}))$.

Clearly, if *(y, z) consists of* **Γ**, then (*y*, *z*) is equal to the convex combination of the *feasible solutions corresponding to the TSP tours in (y, z)* that comprise **Γ**, with weights equal to the associated *flow values*, respectively. Hence, the following proposition shows that (*y*, *z*) is a convex combination of the *feasible solutions corresponding to the TSP tours in (y, z)*.

Proposition 4
Let $(\boldsymbol{y}, \boldsymbol{z}) = (y_{irjkst}, z_{irjupvkst})$ be a feasible solution to *Problem LP*. Let $\mathcal{P}(\boldsymbol{y}, \boldsymbol{z})$ denote the set of all the *TSP tours in (y, z)*.Then, (*y*, *z*) *consists of* $\mathcal{P}(\boldsymbol{y}, \boldsymbol{z})$.

□

Proposition 5
The following statements are true of basic feasible solutions (BFS) of *Problem LP* and TSP tours:

1) Every BFS of *Problem LP* corresponds to a TSP tour;
2) Every TSP tour corresponds to a BFS of *Problem LP*;
3) The mapping of BFS's of *Problem LP* onto TSP tours is surjective.

□

Corollary 1
Problem LP and *Problem IP* (and therefore, *Problem TSP*) are equivalent.

□

Corollary 2
Computational complexity classes *P* and *NP* are equal.

□

3 Numerical Implementation

We used the simplex method implementation of the *OSL* optimization package (IBM) to solve a set of randomly-generated 7-city problems. The travel costs in these randomly-generated problems were taken as uniform integer numbers between 1 and 300. Three of these problems had symmetric costs. The other three randomly-generated problems had asymmetric costs. We also solved an additional set of 7-city problems we refer to as "extreme-symmetry" problems. These "extreme-symmetry" problems are labeled "*xtsp71*," "*xtsp72*," and "*xtsp73*," respectively. In *Problem xtsp71*, all travel costs, t_{ij}, are equal to (-1), except for t_{12} and t_{21} which are equal to 1, respectively. In *Problem xtsp72*, all travel costs, t_{ij}, are equal to 1, except for t_{12} and t_{21} which are equal to (-100), respectively. Finally, in *Problem xtsp73*, all travel costs, t_{ij}, are equal to 0, except for t_{12} and t_{21} which are equal to 1, respectively.

We solved both the dual and primal forms of each of the test problems described above, respectively. Using the dual forms, the averages of the numbers of iterations were 475.0, 1,752.7, and 3,880.5 for the asymmetric, symmetric, and "extreme-symmetry" problems, respectively. The corresponding average computational times were 0.1617, 1.3493, and 9.0785 CPU seconds of Sony VAIO VGN-FE 770G notebook computer (1.8 GHz Intel Core 2 Duo Processor) time, respectively.

For the primal forms, the average number of iterations was 2,203.0, 3,542.0, and 3,315.7 for the asymmetric, symmetric, and "extreme-symmetry" problems, respectively. The corresponding average computational times were 2.8910, 6.5157, and 5.4900 CPU seconds, respectively. The average number of TSP tours examined in the simplex procedure was 1.0, 1.3, and 1.0 for the asymmetric, symmetric, and "extreme-symmetry" problems, respectively.

Overall, we believe our computational experience provided the empirical validation of our theoretical developments in section 2 of this paper that we expected.

4 Conclusions

We have presented a first polynomial-sized linear programming formulation of the TSP. Because the general integer programming problem is *polynomially transformable* to a Hamiltonian Path problem (see Johnson and Papadimitriou [1985, pp. 61-74], our approach can be used to formulate general integer programming problems as linear programs. We believe a key issue for future developments at this point is how to solve TSP's of practical sizes.

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