A Note on Kowalevski Exponents and Polynomial Integrals for Differential Systems

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Abstract: We study an equality between the Kowalevski exponents and the existence of polynomial first integrals for the ABC Lotka-Volterra system.

Key–Words: Ordinary differential equations, Quasi-homogeneous systems, First integrals, Kowalevski exponents, Lotka-Volterra system

1 Introduction

Investigation of first integrals of ODEs is classical work. In the last years much effort has been made to obtain first integrals of 3D dynamical systems. Many of these studies are devoted to the 3D Lotka-Volterra systems [1, 2]. Nevertheless the dynamics of the 3D LV system is far from being understood. In fact, the explicit computation of first integrals is not an easy task. On the other hand, there exists no general method for determining whether or not a given system is integrable. The extreme rarity of integrable dynamical systems makes the quest for them all the most intriguing.

It is known that existence or non-existence of first integrals is related to Kowalevski exponents. Yoshida [3] discussed a necessary condition for existence of first integrals by using Kowalevski exponents. Recently Furta [4] and Goriely [5] show that nonexistence of first integrals is connected with resonance relations among Kowalevski exponents. In this note we study an equality between the Kowalevski exponents and the existence of polynomial first integrals for the ABC Lotka-Volterra system.

2 Preliminaries

Consider an n-dimensional system of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n), \quad i = 1, \cdots, n, \tag{1}$$

with $X_i \in \mathbb{R}[x_1, \cdots, x_n]$. Here $\mathbb{R}[x_1, \cdots, x_n]$ is the ring of polynomials. We denote by

$$X : \mathbb{R}^n \to T\mathbb{R}^n, \quad X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \tag{2}$$

the vector field associated to system (1) by the relations

$$\frac{dx_i}{dt} = X(x_i), \quad i = 1, \cdots, n. \tag{3}$$

Definition 1 *A first integral of the vector field $X : \mathbb{R}^n \to T\mathbb{R}^n$ is a non-constant smooth function $F : \mathbb{R}^n \to \mathbb{R}$ that satisfies the equation*

$$X(F) = \sum_{i=1}^{n} X_i \frac{\partial F}{\partial x_i} = 0. \tag{4}$$

We say that the vector field $X : \mathbb{R}^n \to T\mathbb{R}^n$ is completely integrable if there exist $n-1$ functionally independent almost everywhere first integrals $F_1, F_2, \cdots, F_{n-1}$, i.e., there exist an open set $M \subset \mathbb{R}^n$ whose complement has zero measure, so that $rank(dF_1 \wedge dF_2 \wedge \cdots \wedge dF_{n-1}) = n-1$ for each point in M. In this case its trajectories are determined by intersecting the invariant set $\mathcal{N}_i = \{F_i^{-1}(c_1) | c_i \in \mathbb{R}, \}$ $i = 1, \cdots, n-1$.

Definition 2 *The system (1) is called a quasi-homogeneous system of degree $m \in \mathbf{N}\backslash\{1\}$ with weight exponents $g_1, \cdots, g_n \in \mathbf{Z}\backslash\{0\}$, if for any $\alpha \in \mathbb{R}^+$ all the X_i satisfy the following conditions*

$$X_i(\alpha^{g_1} x_1, \cdots, \alpha^{g_n} x_n) = \alpha^{g_i+m-1} X_i(x_1, \cdots, x_n),$$
$$i = 1, \cdots, n. \tag{5}$$

The quasi-homogeneous system possesses particular solution in the form

$$x_i(t) = a_i t^{-g_i}, \quad i = 1, \cdots, n, \tag{6}$$

where $a_1, \cdots, a_n$ satisfy the equations

$$X_i(a_1, \cdots, a_n) + g_i a_i = 0, \quad i = 1, \cdots, n. \tag{7}$$

Definition 3 *The $n \times n$ matrix*

$$\mathcal{K}_{ij} = \frac{\partial X_i}{\partial x_j}(a_1, \cdots, a_n) + \delta_{ij} g_j,$$
$$i, j = 1, \cdots, n, \tag{8}$$

is called a Kowalevski matrix. Its eigenvalues $\lambda_1, \cdots, \lambda_n$ are the Kowalevski exponents of the system (1) around the solution (6).

Theorem 4 *[4,5] If all Kowalevski exponents $\lambda_1, \lambda_2, \cdots, \lambda_n$ for the system (1) are* **N***-independent, that is, that no resonant condition of the following type*

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \quad k_j \in \mathbf{N} \cup \{0\}, \ \sum_{j=1}^{n} k_j \geq 1$$

is fulfilled, then there is no polynomial first integral.

This Theorem gives a necessary condition for existence of polynomial first integrals for quasi-homogeneous systems of ordinary differential equations.

3 The 3D Lotka-Volterra system

The 3D LV system is defined by the vector field

$$X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y} + X_3 \frac{\partial}{\partial z}, \tag{9}$$

with

$$X_1 = x(Cy + z + \lambda)$$
$$X_2 = y(x + Az + \mu)$$
$$X_3 = z(Bx + y + \nu)$$

This system introduced by Lotka [1] and Volterra [2] has been widely used in a large variety of problems in biology, chemistry, physics, economy etc. As it is well known this system shows a very rich behavior, from complete integrability to chaos (at least numerically), according to the values of the parameters $(A, B, C, \lambda, \mu, \nu)$ that appear in the equations of motion.

We focus on the ABC system, that is, on the 3D LV system with $\lambda = \mu = \nu = 0$. The polynomial first integrals for ABC system has been analyzed by Muolin-Ollagnier [6] using the Darboux theory of integration. Moulin-Ollagnier [6] characterizes all Lotka-Voltera polynomials first integrals as follows:

Theorem 5 *The ABC system possesses polynomial first integrals if and only if one of the following cases holds:*

(i) $ABC = -1$

(ii) $C = -1 - 1/A, A = -1 - 1/B, B = -1 - 1/C$

(iii) $A = -k_3 - 1/B, B = -k_1 - 1/C, C = -k_3 - 1/A$, where up to a permutation, (k_1, k_2, k_3) is one of the triples: (1, 2, 3), (1,2,3), (1,2,4)

The polynomials $x - Cy + ACz$ and $A^2B^2x^2 + y^2 + Az^2 - 2ABxy - 2A^2Bxz - 2Ayz$ are first integrals in case (i) and (ii) respectively. In each of the case (iii), there exists a homogeneous first integral of degree 3, 4 or 6 respectively.

4 Kowalevski exponents and polynomial first integrals

The ABC system

$$\frac{dx}{dt} = x(Cy + z)$$
$$\frac{dy}{dt} = y(x + Az)$$
$$\frac{dz}{dt} = z(Bx + y)$$

is quasi-homogeneous system of degree m=2 with weight exponents $g_1 = g_2 = g_3 = 1$. This system admits the particular solution

$$x = a_1 t^{-1}, \ y = a_2 t^{-1}, \ z = a_3 t^{-1} \tag{10}$$

where constants a_1, a_2, a_3 are determined by

$$Aa_3 + a_1 = -1, \ Ba_1 + a_2 = -1,$$
$$Ca_2 + a_3 = -1. \tag{11}$$

If $\triangle = ABC + 1 \neq 0$, the system (11) has the unique solution

$$a_1 = A(1 + \kappa_1)/\triangle, \ a_2 = B(1 + \kappa_2)/\triangle$$
$$a_3 = C(1 + \kappa_3)/\triangle \tag{12}$$

where

$$\kappa_1 = -C - 1/A, \kappa_2 = -A - 1/B, \kappa_3 = -B - 1/C$$

The Kowalevski matrix reads

$$\mathcal{K} = \begin{pmatrix} 0 & Ca_1 & a_1 \\ a_2 & 0 & Aa_2 \\ Ba_3 & a_3 & 0 \end{pmatrix}.$$

The corresponding Kowalevski exponents are defined by the characteristic equation

$$\lambda^3 - \lambda(Aa_2a_3 + Ba_1a_3 + Ca_1a_2) - \triangle a_1a_2a_3 = 0. \quad (13)$$

Theorem 6 *If*

$$\frac{1}{1+\kappa_1} + \frac{1}{1+\kappa_2} + \frac{1}{1+\kappa_3} - 1 \neq \frac{1}{n(n-1)}, \quad n \in \mathbf{N}\backslash\{1\}.$$

then the ABC Lotka-Volterra system does not possess a polynomial first integral.

Proof: First, we note that - 1 is always a Kowalevski exponent (see Refs. [3, 4]). It is easy to show that

$$\triangle a_1a_2a_3 = Aa_2a_3 + Ba_1a_3 + Ca_1a_2 - 1$$

and Eq. (13) can equivalently be expressed as

$$(\lambda+1)(\lambda^2 - \lambda - \triangle a_1a_2a_3) = 0.$$

Hence the Kowalevski exponents are readily obtained as

$$\lambda_1 = -1, \ \lambda_{2;3} = \frac{1}{2}\left(1 \mp \sqrt{1 + 4\triangle a_1a_2a_3}\right). \quad (14)$$

Thus, $\lambda_2, \lambda_3 \in \mathbf{Z}\backslash\{0\}$ if and only if

$$1 + 4\triangle a_1a_2a_3 = (2n-1)^2, n \in \mathbf{N}\backslash\{1\}.$$

Since

$$Aa_2a_3 + Ba_1a_3 + Ca_1a_2 = \triangle a_1a_2q_3\left(\frac{1}{1+\kappa_1} + \frac{1}{1+\kappa_2} + \frac{1}{1+\kappa_3}\right)$$

we have

$$\frac{1}{n(n-1)} = \frac{1}{1+\kappa_1} + \frac{1}{1+\kappa_2} + \frac{1}{1+\kappa_3} - 1. \quad (15)$$

Therefore, according to Theorem 4, the ABC LV system has no polynomial integral, if $k_1 \neq k_2\lambda_2 + k_3\lambda_3$, $k_1, k_2, k_3 \in \mathbf{N} \cup \{0\}$ and $k_1 + k_2 + k_3 \geq 1$. That is, if the Kowalevski exponents $\lambda_2, \lambda_3 \notin \mathbf{Z}\backslash\{0\}$. This completes the proof.

Theorem 7 *The ABC system possesses a polynomial first integral of degree $n \geq 2$ if and only if*

- $\lambda_1 = -1, \ \lambda_2 = 1-n, \ \lambda_3 = n$
- $n = 2, 3, 4, 6.$

Proof: The proof immediately follows from Theorems 5 and 6. The first condition of the Theorem is the necessary condition given by Theorem 6. Indeed, because of (14): $\lambda_3 = n, \ n \in \mathbf{N}\backslash\{1\}$, and the Kowalevski exponents read

$$\lambda_1 = -1, \ \lambda_2 = 1-n, \ \lambda_3 = n. \quad (16)$$

The second condition defines the sufficient condition for the existence of a polynomial integral for the ABC LV system. The equation (15), under the conditions (ii) and (iii) of Theorem 5, has only the following solutions

- $n = 2 \quad (\kappa_1, \kappa_2, \kappa_3) = (1, 1, 1)$
- $n = 3 \quad (\kappa_1, \kappa_2, \kappa_3) = (1, 2, 2)$
- $n = 4 \quad (\kappa_1, \kappa_2, \kappa_3) = (1, 2, 3)$
- $n = 6 \quad (\kappa_1, \kappa_2, \kappa_3) = (1, 2, 4).$

Thus the ABC LV system has polynomial first integrals of degree $n = 2, 3, 4, 6$, and the corresponding Kowalevski exponents $(\lambda_1, \lambda_2, \lambda_3)$ are defined as follows

- $n = 2 \quad (\lambda_1, \lambda_2, \lambda_3) = (-1, -1, 2)$
- $n = 3 \quad (\lambda_1, \lambda_2, \lambda_3) = (-1, -2, 3)$
- $n = 4 \quad (\lambda_1, \lambda_2, \lambda_3) = (-1, -3, 4)$
- $n = 6 \quad (\lambda_1, \lambda_2, \lambda_3) = (-1, -5, 6)$

We note that the Kowalevski exponent λ_3 is equal to the degree n of the polynomial integral. This ends the proof.

5 Conclusion

In this note we do not discuss the so-called Painlevé analysis where the analysis of local solutions around the singularities is the main tool to test for integrability. We focus on some results connecting directly with the Kowalevski exponents to the existence of polynomial first integrals. In this paper we test predictive power of the theory of the Kowalevski exponents.

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