Box-counting dimension of the angular superposition from two Cantor grids

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Abstract: - In the present work we study the fractality of the structure obtained when two Cantor grids are superimposed for different relative angular displacements and, as a special case, the moiré structures obtained either in the complete structure as in the normalized fringe profiles. We use the box-counting technique for the case of two grids with arbitrary fractal dimensions.

Key-Words: - Image processing, optical signal processing, moiré effect, Cantor grids, box-counting

1 Introduction

The study of the moiré effect and its applications are an important area of the applied optics, and a wide bibliography exists in this respect [1-4]. The superposition of two periodic grids (with their corresponding parameters) is shown in Fig. 1. The fringes obtained for the case of angular moiré (with period D_M) as well as the fringe profiles, as a measure of the correlation between two grids, are characteristics included in the studies developed for the patterns of fringes obtained in different cases.

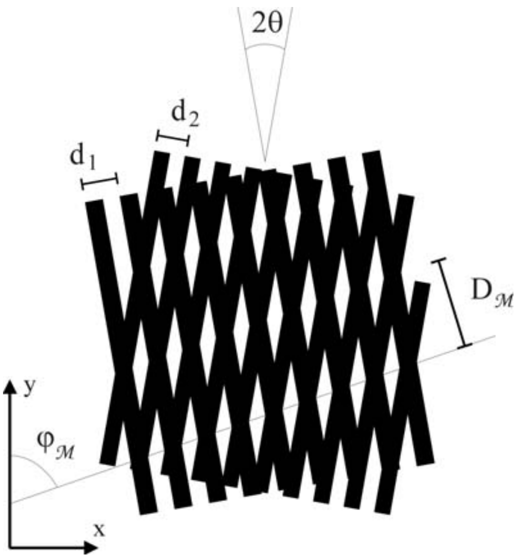


Figure 1 – Moiré effect from two periodic grids, with periods d_1 and d_2, and angular displacement 2θ.

Here, we are interested in the fractality of the moiré structure and the normalized profiles which appear in the angular superposition of gratings with fractal characteristics. As in previous works [5,6], these gratings are built with a product of periodic components. In the moiré superposition, such periodic grids become important for obtaining the parameters of the moiré structure: period and angular direction.

2 Moiré from Cantor grids

When two Cantor grids, with an angular difference between them, are superimposed, we can obtain moiré structures. In this work we use Cantor grids included in an initial periodic structure (see Fig. 2). The theory for the construction of such grids has been presented in previous works [5,6].

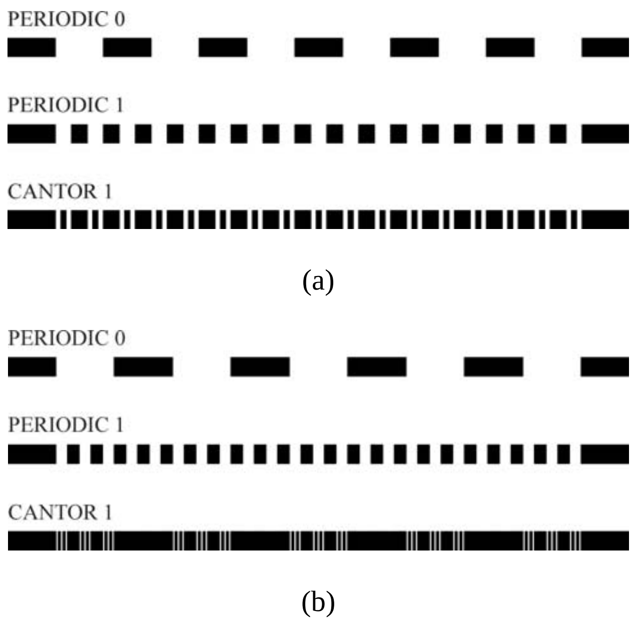
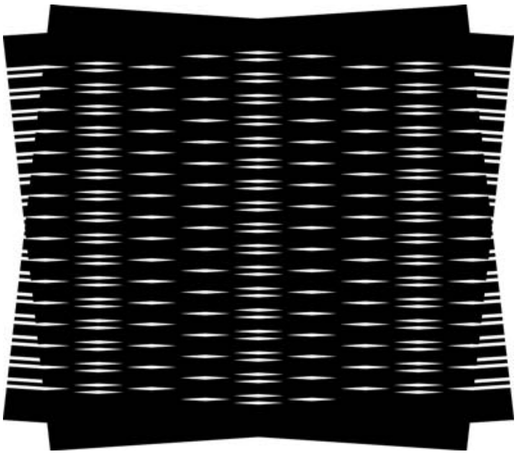
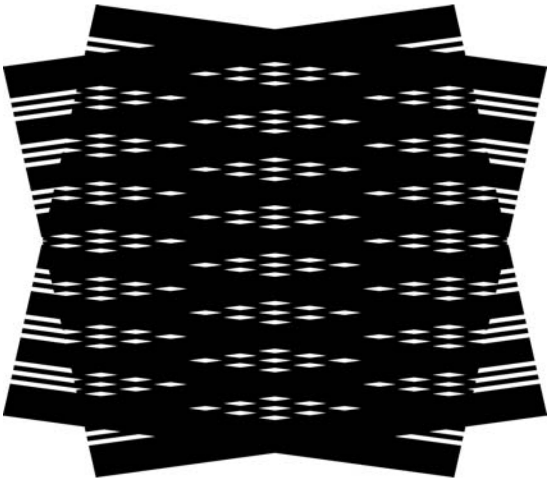


Figure 2 – Cantor grids obtained with the product of periodic components for fractal dimensions: (a) $D \approx 0.6309$, (b) $D \approx 0.6826$.



(a)



(b)

Figure 3 – Superposition of each Cantor grids from Fig. 1, which shows the moiré structure.

Also, we have demonstrated that, similarly to the case of periodic grids [2], the moiré obtained when two Cantor grids are superimposed, can be written as the product [7-9]:

$$M_{12}(x,y)=\prod_{k=0}^{N}\left\{\sum_{n_k=-\infty}^{+\infty}\sum_{m_k=-\infty}^{+\infty}C_{n_k}\,C_{m_k}\right. \qquad (1)$$
$$\left.exp\left[2\pi i\left(\frac{s_1^k\,n_k}{d_1}\Phi_1(x,y)+\frac{s_2^k\,m_k}{d_2}\Phi_2(x,y)\right)\right]\right\},$$

where s_1, s_2 are the scaling factors, d_1, d_2 are the initial periods and we defined:

$$\Phi_1(x,y)=x\cos\theta+y\sin\theta\,, \qquad (2)$$
$$\Phi_2(x,y)=x\cos\theta-y\sin\theta\,.$$

Then, the moiré fringes are obtained when:

$$\frac{s_1^k\,n_k}{d_1}\Phi_1(x,y)+\frac{s_2^k\,m_k}{d_2}\Phi_2(x,y)=p\,, \qquad (3)$$

and the parameters (period and angular direction) of such moiré fringes are given by:

$$D_{\mathcal{M}k}=\frac{d_{10}\,d_{20}}{\sqrt{s_1^{2k}\,d_{1k}^2+s_2^{2k}\,d_{2k}^2-2s_1^k\,s_2^k\,d_{1k}\,d_{2k}\cos\theta}}\,, \qquad (4)$$
$$\sin\varphi_{\mathcal{M}k}=\frac{(d_{1k}+d_{2k})\sin\theta}{\sqrt{s_1^{2k}\,d_{1k}^2+s_2^{2k}\,d_{2k}^2-2s_1^k\,s_2^k\,d_{1k}\,d_{2k}\cos\theta}}\,.$$

In these equations we can observe that the properties of the moiré effect obtained with Cantor grids are characterized from the periodic components contained in the fractal structure.

The superposition of two identical Cantor grids, with an angular displacement among them, is shown in Fig. 3, for both cases considered in Fig. 2. Starting from Eq. (3), it can be seen that the periods and the angular direction of the moiré fringes also have a relation on the scaling, and not only on the periods of each periodic component.

3 Box-counting dimension of the superposition

The box-counting dimension is defined as:

$$D=-\lim_{\delta\to 0}\frac{log(N)}{log(\delta)} \qquad (5)$$

being N the number of boxes with size δ.

Starting from structures with a degree of fractality along one dimension, structure with fractality in two dimensions can be obtained, by means of an angular superposition of the same ones, using the product operation previously used. In this section, the mathematical expression for the superposition of two Cantor fractals when an angular displacement between them is taken into account is obtained, around a point of the x-y plane, where they are contained. In the first place, emphasize is made on a theorem referred to the intersection between two fractal sets [10] that will be useful for the construction that we want to make.

Theorem 1. Let $\boldsymbol{A}^k, \boldsymbol{B}^k \subset \boldsymbol{R}^n$ be Borel sets, and let $\boldsymbol{G}$ be a group of transformations on $\boldsymbol{R}^n$. Then:

$$\dim_H \left[\boldsymbol{A}^k \cap \Gamma\left(\boldsymbol{B}^k\right)\right] \geq \dim_H \left[\boldsymbol{A}^k\right] + \dim_H \left[\Gamma\left(\boldsymbol{B}^k\right)\right] - n \tag{6}$$

for a set of motions $\Gamma(\boldsymbol{G})$ of positive measure in the following cases:

a) $\boldsymbol{G}$ is the group of similarities and $\boldsymbol{A}^k$ and $\boldsymbol{B}^k$ are arbitrary sets.

b) $\boldsymbol{G}$ is the group of rigid motions, $\boldsymbol{A}^k$ is arbitrary and $\boldsymbol{B}^k$ is a rectifiable curve, surface, or manifold.

c) $\boldsymbol{G}$ is the group of rigid motions and $\boldsymbol{A}^k$ and $\boldsymbol{B}^k$ are arbitrary, with either:

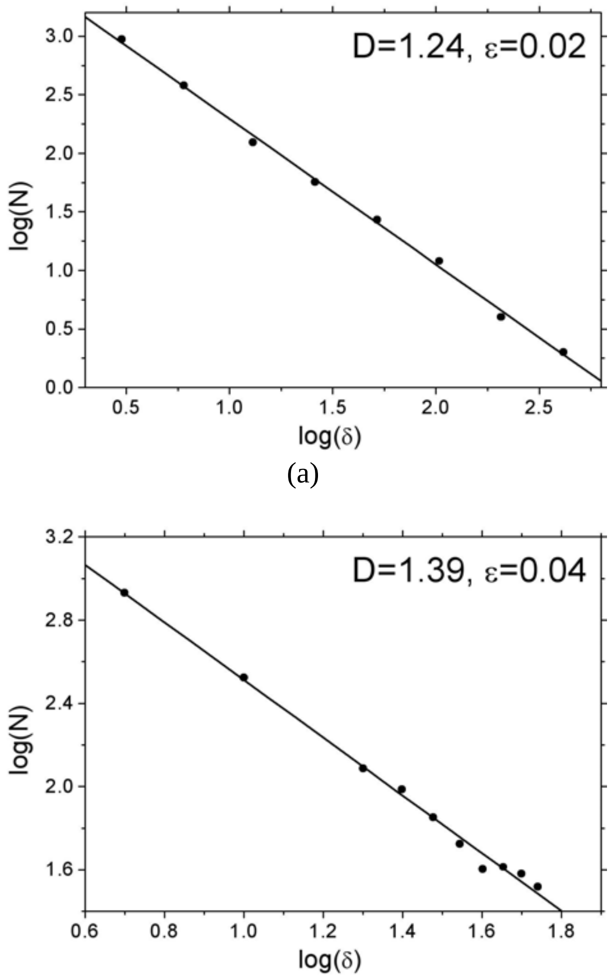
$$\dim_H \left[\boldsymbol{A}^k\right] > \frac{1}{2}(n+1), \ \dim_H \left[\boldsymbol{B}^k\right] > \frac{1}{2}(n+1) \tag{7}$$

From this theorem, it is possible to demonstrate that the fractal dimension of two Cantor structures is given as the cartesian product [10], and then:

$$\begin{aligned} \dim_H (\boldsymbol{A} \times \boldsymbol{B}) &= \dim_H \boldsymbol{A} + \dim_H \boldsymbol{B} \quad , \\ \boldsymbol{A} = \boldsymbol{B} \Rightarrow \dim_H (\boldsymbol{A} \times \boldsymbol{A}) &= 2 \dim_H \boldsymbol{A} \quad , \end{aligned} \tag{8}$$

Because the several definitions of dimension give the same value for the case of Cantor sets [11], the previous results can be synthesized with the numeric calculation of one of them, for example box-counting. Next, we calculate the box-counting dimension for the examples included in Fig. 3, when the angle is 10 degrees.

The results show the fractality in such structures which is clearly dependent of the order in the grids considered, and give results in accordance with the expected results from Eqs. (8), considering the error that arises of the regression method used in each superposition.



(a)

(b)

Figure 4 – Box-counting dimension for the superposition of two identical grids: (a) $D \approx 0.6309$, (b) $D \approx 0.6826$.

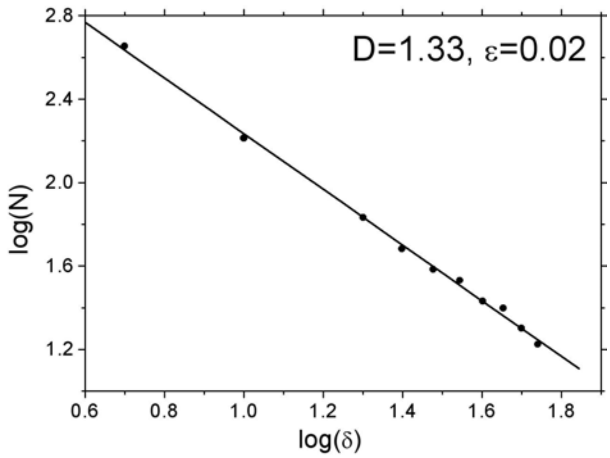


Figure 5 – Box-counting dimension for the superposition of Cantor grids with $Df \approx 0.6309$ and $Df \approx 0.6826$.

3.1 Evolution of the box-counting dimension as a function of the relative angular displacement

Since measurements over the fractal structure must be done over a finite dimension we study the evolution of the fractal dimension for different rotation angles between both Cantor structures. First, we establish a fixed (and finite) window, over which the box-counting dimension is performed. When one grid is rotated over the other, inside the chosen window, different structures appear. If both grids have the same fractal structure, it is logical to find an initial dimension very similar to the one determined by Eq. (5) for each grid, but the window determines a finite size. Then, there is an evolution towards the value of the cartesian product obtained by the same Eq. (8). Such evolution is shown in Fig. 6, which has been approximated trough an exponential decay of first order. We can see that, for small angles, the moiré fringes visualized in the structure (see Fig. 3) is related with a bigger value of the box-counting dimension. This characteristic is present until 10 to 20 degrees, and it is a way for a measurement of the limit in the moiré effect for the superposition of two fractal structures.

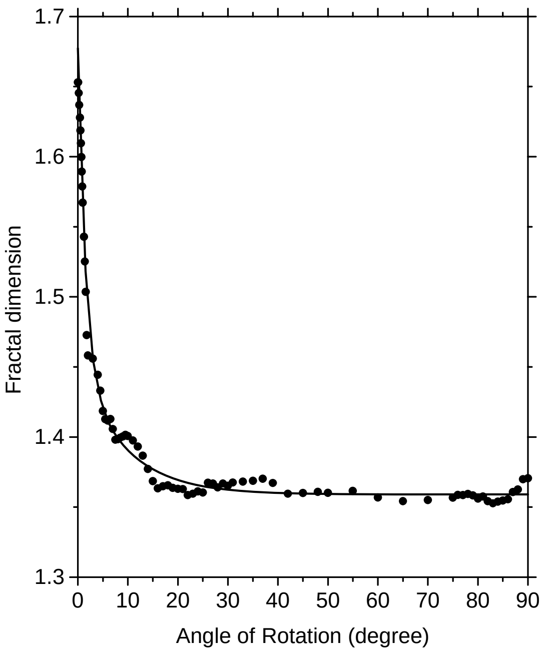


Figure 6 – Evolution of the box-counting dimension with the angle of rotation between two Cantor grids.

3.2 Dimensional analysis of the normalized moiré profiles

Now, we study the structure of the moiré profiles. For this purpose we make the change of coordinates $(x,y)\rightarrow(u,v)$ and $M_T^N(x,y)\rightarrow M_T^N(u,v)$, where v is the coordinate along the moiré fringes and u is the corresponding perpendicular coordinate. The intensity values obtained for the moiré fringes are denoted by the mean value [1]:

$$F^N(u)=\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{+T}M_T^N(u,v)\,dv \qquad \Rightarrow F^N(M_2)=\frac{1}{M_2}\sum_{j=1}^{M_1}I(M_1,j) \tag{9}$$

being T a displacement along the v-axis.

For the case of images from the moiré structures here obtained, which have a finite number of pixels, the integral of Eq. (9) can be approximated as the finite sum. In this case, M_2 indicates the number of pixels along the fringes of the moiré structure, M_1 is the corresponding number of pixels for the perpedicular direction and $I(i,j)$ is the intensity level registered at the pixel (i,j) in the image of the moiré structure.

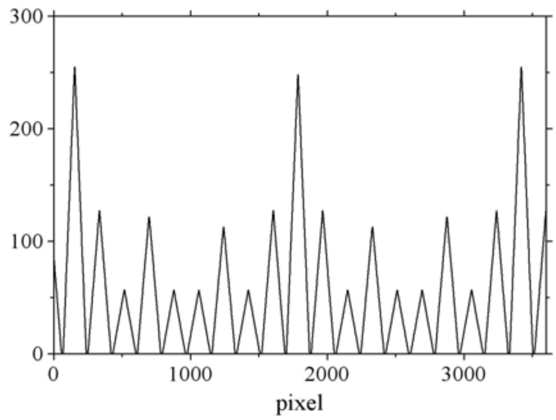


Figure 7 – Example of moiré profile for the case $Df\approx 0.6309$.

The trends of the normalized fringe profiles occupy the plane, and for this reason have high values in the box-counting dimension. In the case of $D\approx 0.6309$ the secondary picks of intensity are bigger, compared with the case $D\approx 0.6826$ and then, it has a bigger value in the box-counting.

4 Conclusion

We have seen that the structures obtained from the angular superposition of two periodic Cantor grids form fractal structures with periodicity. For small angles, the moiré effect is present. The box-counting method is applicable in these cases for determining the fractal dimension for the superposition of two Cantor grids. This is achieved by using the box-counting method and the result from the cartesian product of two Cantor structures (Eqs. (5)). This method is applied to different cases: 1) for the total structure at a certain angular displacement, 2) to studying the evolution of dimension with the angular displacement, 3) for the normalizad profiles. Each case is very important to establish different characteristics of the obtained structures, involving a relation between the moiré structure and the angular displacement. We expect to develop different applications in the future to relate moiré effect and the fractal dimension of complex and fractal grids.

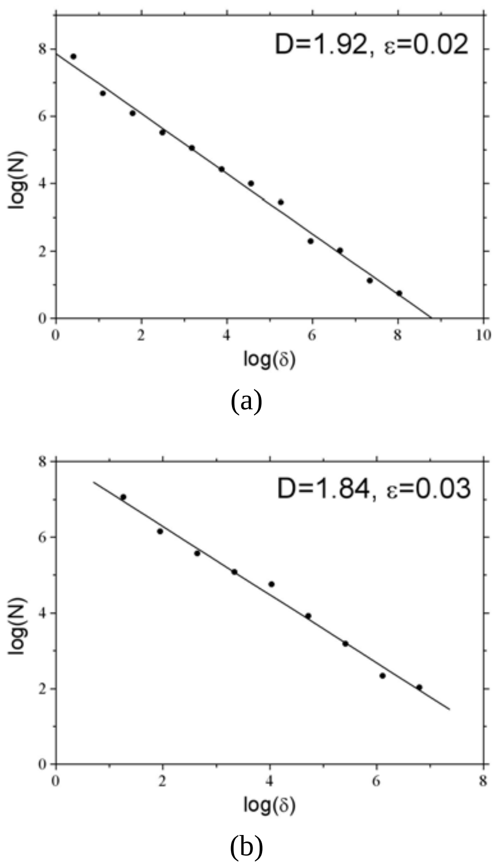


Figure 8 – Box-counting dimension for the normalized moiré profiles of the structures in Fig. 2.

Acknowledgments

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