# A topological order for a rectangular covering problem

DANIELA MARINESCU
Transilvania University of Braşov
Department of Computer Science
Iuliu Maniu 50, 500091 Brasov
ROMANIA

PAUL IACOB
Transilvania University of Braşov
Department of Computer Science
Iuliu Maniu 50, 500091 Brasov
ROMANIA

KINGA KISS-IAKAB
Transilvania University of Braşov
Department of Computer Science
Iuliu Maniu 50, 500091 Brasov
ROMANIA

*Abstract:* We consider a rectangular covering problem, where a rectangular support $\mathcal{P}$ is covered by *k* rectangular pieces $C_i$, for *i=1, 2, ..., k*, without gaps or overlapping. We intend to cover the support $\mathcal{P}$ with the *k* pieces in a certain order so that, at the moment of the placement of the piece $C_i$, the northern and western borders of this piece are completely covered. Starting from a graph representation of a covering model we prove that this order is a topological order. We present a kind of topological sorting for this problem, of linear complexity. At the end we present some practical applications of this topological order.

*Key–Words:* covering problem, topological order

## 1 Introduction

The covering problems appear in many fields starting from mathematical problems up to practical problems of cutting-covering [4], problems of pattern recognition [5], or problems of communications [9]. On the other hand, the topological sort of a set of elements is an often treated problem, details can be found in introductory algorithm literatures [2] [3]. The problem is more complicated if the elements of the set are two-dimensional and there are more sorting criteria which must be fulfilled simultaneously (see for example [1], where the elements are triangles). We are dealing in this paper with the following ordering of a rectangular covering problem: starting from the covering model, (that can be obtained for example as in [4]), which will be the order of the covering pieces for the placement on the support? In this order we are taking into account that at the moment of the placement of a piece $C_i$, the northern and western borders of this piece are completely covered. Also we take as starting point for covering procces the N-W corner of the support and for ending point the S-E corner. The problem can be extended by choosing another starting, respectively ending point, depending on the technological restrictions of some practical problems.

Let $\mathcal{P}$, a rectangular plate, characterized by length *l* and width *w*. The plate $\mathcal{P}$ is covered with *k* rectangular pieces, $C_i$ , *i =1, 2, ..., k*, characterized by length $l_i$ and width $w_i$.

**Definition 1** *A rectangular covering model is an arrangement of the k rectangular components $C_i$ on the supporting plate $\mathcal{P}$, so that $\mathcal{P}$ is completely covered by the components $C_i$, without gaps or overlapping.*

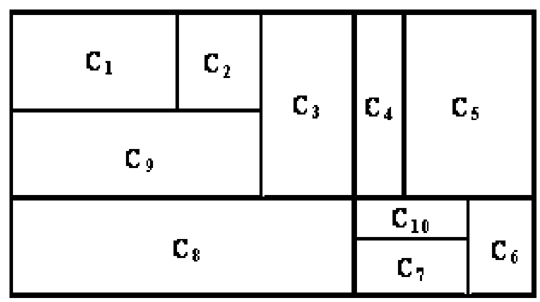


Figure 1: A rectangular covering model

*Example 1.* Let the covering model from Figure 1. where $\mathcal{P}$(235x164), $C_1$(75x57), $C_2$(37x57), $C_3$(42x107), $C_4$(22x107), $C_5$(59x107), $C_6$(30x57), $C_7$(51x33), $C_8$(154x57), $C_9$(112x50), $C_{10}$(51x24).

In the set of the rectangles $\{C_1, C_2, \ , C_k\}$ from the covering model we define a downwards adjacency relation and a rightwards adjacency relation.

**Definition 2** *The rectangle $C_i$ is downward adjacent with rectangle $C_j$ if in the covering model, $C_j$ is to be found downward $C_i$ and their borders have at least two common points.*

**Definition 3** *The rectangle $C_i$ is rightward adjacent with rectangle $C_j$ if in the covering model, $C_j$ is to be found rightward $C_i$ and their borders have at least two common points.*

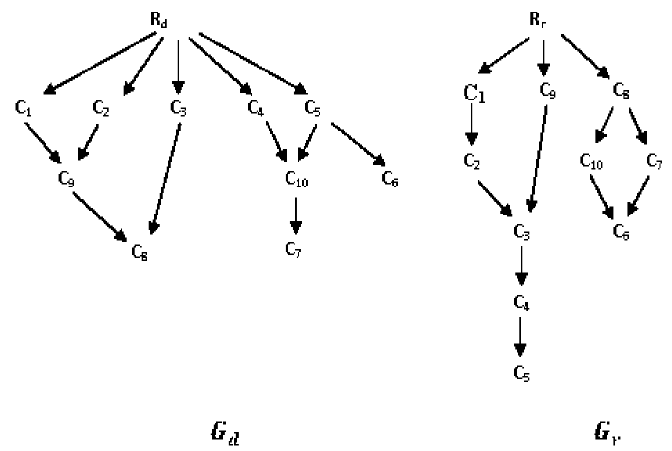


Figure 2: Graphs $G_d$ and $G_r$

Let $C$ = { $C_1$ , $C_2$, ..., $C_k$ } and $R_d$, $R_r \notin C$. For any covering model, we can define a graph of *downwards adjacency*, $G_d$, and another one of *rightwards adjacency*, $G_r$.

**Definition 4** *[6] The graph of downward adjacency $G_d$ = (C∪{$R_d$}, $\Gamma_d$) has as vertices the rectangles $C_1$, $C_2$, ..., $C_k$ and a new vertex $R_d$ symbolizing the northern borderline of the supporting plate P. $\Gamma_d(C_i)$ = $C_j$ if and only if $C_i$ is downward adjacent with $C_j$. $\Gamma_d(R_d)$ = $C_j$ if and only if there is no i∈{1, 2, ..., k} so that $C_i$ is downward adjacent with $C_j$ (the vertices without ascendants of the subgraph $G_d$ = (C, $\Gamma_d$) are connected to the vertex $R_d$ through an arch from vertex $R_d$).*

**Definition 5** *[6] The graph of rightward adjacency $G_r$ = (C∪{$R_r$}, $\Gamma_r$), where $R_r$ symbolizes the western border. $\Gamma_r(C_i)$ = $C_j$ if and only if $C_i$ is rightward adjacent with $C_j$. $\Gamma_r(R_d)$ = $C_j$ if and only if there is no i∈{1, 2, ..., k} so that $C_i$ is rightward adjacent with $C_j$ (the vertices without ascendants of the subgraph $G_r$ = (C, $\Gamma_r$) are connected to the vertex $R_r$ through an arch from vertex $R_r$).*

*Example 2.* Let the covering model from Figure 1. The graphs $G_d$ and $G_r$, are represented in Figure 2.

We remark that in the graphs $G_d$ and $G_r$ the vertex $R_d$ (respectively $R_r$) is connected by means of an arch of vertex $C_i$ if and only if $C_i$ touches the northern (respectively the western) border of the support $\mathcal{P}$.

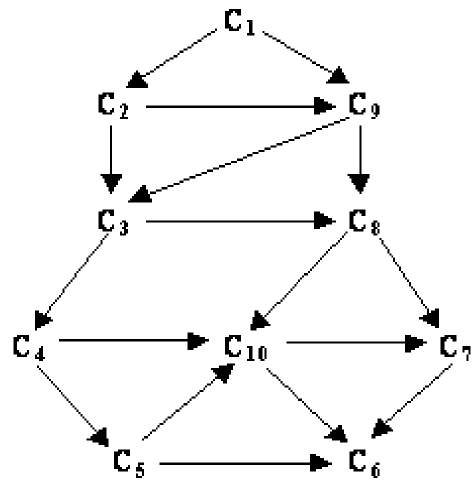


Figure 3: The compound graph

## 2 Compound Graph for the Covering Model

Due to the properties of the graphs $G_d$ and $G_r$ proved in [6], it is possible to represent simultaneously these graphs by a single trivalent adjacency matrix, *T*, a matrix with elements in {0, 1, 2} (see [8]). It results that we can build a compound graph from $G_d$ and $G_r$ as it follows:

**Definition 6** *For any covering model, we define the graph of compound adjacency, $G_c$ = (C, $\Gamma_c$), where $\Gamma_c(C_i)$ = $C_j$ if and only if $\Gamma_d(C_i)$ = $C_j$ or $\Gamma_r(C_i)$ = $C_j$.*

The adjacency matrix, *T'*, for $G_c$ is similar to the trivalent matrix *T* [7, 8], where we omit the first two columns and lines (corresponding to vertices $R_d$ and $R_r$) and we replace 2 by 1.

*Example 3.* Let the covering model from Figure 1. The compound graph $G_c$ is presented in the Figure 3.

We consider in the following that every arch in the compound graph $G_c$ has the value 1.

**Definition 7** *The length of paths in the compound graph is defined by the function $p : C \to \mathscr{P}(\mathbb{R}^+)$ where p(E) is the set formed with the lengths of the paths from the northwestern corner to the vertex E.*

So the length of a path in the compound graph is the number of its arches.

**Definition 8** *The longest path in the compund graph is defined by the function $lp : C \to \mathbb{R}^+$ , where lp(E) = max{ x | x ∈ p(E) }, where E ∈ C.*

**Theorem 9** *For any rectangular covering model, the attached compound graph $G_c$ is acyclic.*

**Proof:** The graph $G_c$ is composed of two acyclic graphs, $G_d$ and $G_r$, which represent the covering model. In [8] it is proved that these two graphs do not overlap so that if there is an arch from $C_i$ to $C_j$ in $G_d$ or $G_r$ then there is no arch from $C_i$ to $C_j$ in $G_d$ ( respectively $G_r$) nor from $C_j$ to $C_i$. In addition, this property is also extended to the paths in the two graphs: if there is a path from $C_i$ to $C_j$ in $G_d$ or $G_r$ then:

- there is no path from $C_j$ to $C_i$ in $G_d$ (respectively $G_r$);
- there is no path from $C_i$ to $C_j$ in $G_r$ (respectively $G_d$);
- there is no path from $C_j$ to $C_i$ in $G_r$ (respectively $G_d$).

We presume that by the compounding of the two graphs $G_d$ and $G_r$ we obtain a cyclic graph $G_c$. In this case there is a simple path $\mu$ in $G_c$, which leaves from an element $S_i$ and it returns to $S_i$. This path can be:

- a path from $G_d$, but then $\mu$ can't return to $S_i$ because $G_d$ is acyclic;
- a path from $G_r$, but then $\mu$ can't return to $S_i$ because $G_r$ is acyclic;
- a path compound of paths from $G_d$ alternating with paths from $G_r$.

In this last case, we presume that the path $\mu$ begins with an arch from $G_d$ and has the form $(\mu_1, \mu_2, ..., \mu_n)$, where $\mu_{2i+1}$ is a path in $G_d$ and $\mu_{2i}$ is a path in $G_r$. Then the path $\mu_1$ in $G_d$ starts with $S_i$ and goes downwards (to the left or to the right) to a vertex $S_j$. Then $\mu$ continues with the path $\mu_2$ in $G_r$ which goes rightwards (up or down) from $S_j$ to $S_k$.

As it is proved in [8] the path $\mu_2$ does not cross $\mu_1$ (except $S_j$) and is rightwards to $\mu_1$, so it does not contain $S_i$. Continuing in the same way it results that no one of the paths $\mu_2, \mu_3, ..., \mu_n$ can contain $S_i$. So the presumption that $G_c$ is cyclic is false. □

**Definition 10** *[2] A topological sorting of a directed acyclic graph G = (C, $\Gamma$) is a linear ordering of all its vertices so that, if G contains an arch ($C_i$, $C_j$) then $C_i$ appears in the order before $C_j$.*

**Theorem 11** *There is a topological order of the vertices from the set C in the compound graph $G_c$.*

**Proof:** Applying Theorem 9 and the results from [3], it results that there is a topological order of the vertices from *C*, because the compound graph $G_c$ is acyclic. □

Due to the Definition 6 of the graph $G_c$, a topological order of the vertices from the set *C* means that if there is an arch from $C_i$ to $C_j$ in $G_c$ i.e. if $C_j$ is on the right of $C_i$ or under $C_i$ and has two common points with $C_i$ then $C_i$ appears in the order before $C_j$. Returning to the significance of the compound graph $G_c$ for a covering model, it results that an element $C_j$ appears in the order after all the elements from the western and northern border of $C_j$.

**Theorem 12** *The compound graph for a rectangular covering model $G_c$ is a particular transport network, where there is a single vertex without ascendants and there is a single vertex without descendants.*

**Proof:** Let $S_1, S_2, ..., S_k$ the topological order of *C*. In [8] it is proved that there is only one vertex $C_i \in \Gamma_d\ (R_d) \cap \Gamma_r\ (R_r)$, i.e. a single vertex without ascendants in the graph $G_c$ and this element is the northwestern corner. Of course this element is $S_1$, the first in the topological order. Also there is only one $C_i$ without descendants in $G_c$ and this element is the southeastern corner, $S_k$ in the topological order. In addition for every $S_i \in C$ there is a path from $S_1$ to $S_k$, which goes through $S_i$. □

# 3 Topological Sorting Algorithm

We can use the algorithm for topological sorting from [3] or a new algorithm, OVERDIAG, based on our particular network $G_c$.

## 3.1 OVERDIAG Algorithm

We proved in 2 that there is a topological order in $G_c$. Then the adjacency matrix *T'*, of $G_c$, where the vertices are in topological order, is an over diagonal matrix with the main diagonal equal to 0.

We will base our algorithm on two observations:

1. By changing lines and columns in the adjacency matrix, the number of elements equal to zero remains unchanged;
2. We can always find a column with the necessary number of zeroes.

If *T'* is a matrix of dimension *k* x *k* then we have the next algorithm for transformation of the matrix in an over diagonal matrix by changing at the same time a line and a column.

```
OVERDIAG(T',k)
  for i = 1 to k - 1
  { looking for the column containing k - i + 1 zeros,
    in the submatrix T'[r, j] (r = i, ..., k; j = i, ..., k) }
     for j = i to k
        num = 0
        for r = i to k
           if (T'[r, j] = 0) then
              num = num + 1
           endif
        endfor
        if (num = k - i + 1) then
           jfix = j
           break
     endfor
     { changing line[jfix] with line[i]
       and column[jfix] with column[i] }
     for j = i to k
        T'[jfix, j] ↔ T'[i, j]
     endfor
     for r = i to k
        T'[r, jfix] ↔ T'[r, i]
     endfor
     { now we have to fix two corners of the rectangle }
     T'[jfix, jfix] ↔ T'[i, i]
  endfor
return
```

### 3.1.1 Correctness and Complexity

Applying the OVERDIAG algorithm we change the order of the vertices $C_i$ so that the adjacency matrix *T'* for the compound graph $G_c$ became an over diagonal matrix. It follows that $T'_{ij} = 0$ for all $i \geq j$ and it is possible to have $T'_{ij} \neq 0$ only if $i < j$. For the compound graph $G_c$ that means there is an arch from $C_i$ to $C_j$ only if $i < j$ so $C_i$ appears before $C_j$ in the ordered set *C*. It follows that *C* is topologically sorted.

Remark that the OVERDIAG algorithm is linear in $k^2$ - the maximal number of edges in the compound graph $G_c$.

## 4 Conclusions

Rectangular covering problems appear in many domains: covering of a rectangular surface with material: linoleum, carpet [4], iron plate, glass etc.

A problem here, after the determination of a covering model, is to determine the order in which the pieces have to be placed (glued) on the support. An order like this presumes that a piece from the middle of the surface is not placed until we do not "get" to this element. This kind of order can be the topological order given by us, where the placement begins with the northwestern corner of the surface and it ends with the southeastern corner. An element $C_i$ is glued on the support only if its western and northern borders are already glued.

We intend to extend these results by considering an intial starting point and an ending point, which differ from the northwestern corner, respectively from the southeastern corner, and which depend on certain technological conditions.

*References:*


[1] Chi-Wing Fu, Tien-Tsin Wong, Pheng-Ann Heng: Triangle-based view Interpolation without depth-buffering, Journal of Graphics Tools, Vol. 3, Issue 4, 1998, pp.13-31.

[2] Ciurea, E., Ciupala, L.: Algoritmi - Introducere in algoritmica fluxurilor in retele. Ed. Matrix ROM Bucureşti 2006.

[3] Cormen, T.H., Leiserson, C.E., Rivest, R.R.,: Introduction to Algorithms. MIT Press, 1990.

[4] Iacob, P., Marinescu, D., Luca, C.: L-Shape room. Proceeding of WMSCI Orlando Florida USA 2005 Vol III pp. 175-179.

[5] Godoy-Calderon, S., Batiz, J.D., Lazo-Cortez, M.: A non-Classical View of Coverings and its Implications in the Formalization of Pattern Recognition Problems, Proc. of WSEAS Conference, 2003, paper 459-151.

[6] Marinescu, D.: Graphs attached to a rectangular cutting-stock model (French), Itinerant Seminar of Functional Equation, Approximation and Convexity Cluj-Napoca 1988 Preprint No. 6 pp. 209-212.

[7] Marinescu, D.: Properties of the matrices attached to a rectangular cutting-stock model (French), Buletin of the Transilvania University of Braşov seria C Vol XXXIV pp 41-48.

[8] Marinescu, D.: A representation problem for a rectangular cutting-stock model, Foundations of Computing and Decision Sciences Poznan Poland (to appear).

[9] Paschos, S.A., Paschos,V.Th., Zissimopoulos, V.: A model to approximately cover an area by antennas, Proc. of WSEAS Conference, 2003, paper 458-187.