Approximation Properties of a Sequence of Linear and Positive Operators

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Abstract: In this paper we study the techniques of linear combinations starting from the studies made by H. Bohman (1952), P.L.Butzer (1953; [2]), (P.P. Korovkin (1953); [11]), T. Popovici (1959; [13]), D.D. Stancu ([15]) respectively the results obtained by E. W. Cheney and A. Sharma [3], S. Eisenberg and B. Wood [16], M. Frenţiu [5], A. Lupaş [9], [10], R. Martini [12]. We define the linear combinations for Favard-Szász S_n operators we obtain different estimation of the remainder for $S_n^{[2k]}$ operator.

Key–Words: Approximation Theory, Linear Positive Operators. Typing manuscripts, LaTeX

1 Introduction

The Favard - Szász operators are de£ned by

$$(S_nf)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

$(n = 1, 2, ...,)$. If further f is twice differentiable at x, there holds the asymptotic relation

$$\lim_{n\to\infty} n[(S_nf)(x) - f(x)] = \frac{x}{2}f''(x).$$

For their proofs etc. see E. W. Cheney and A. Sharma [3], S. Eisenberg and B. Wood [4], M. Frenţiu [5], A. Lupaş [7], [8], R. Martini [12], B. Wood [16].

We derive a few basic formulae about Favard - Szász operators. We define

$$\delta_{n,r} = e^{-nx}\sum_{k=0}^{\infty}\left(\frac{k}{n} - x\right)^r\frac{(nx)^k}{k!}$$

and $m_{n,r} = n^r\delta_{n,r}$, $r = 0, 1, 2, ...$

We have

$$\delta_{n,m} = \frac{x}{n^{m-1}}\sum_{k=0}^{m-2}\binom{m-1}{k}n^k\delta_{n,k}. \quad (1)$$

Now

$$\delta'_{n,m} = -n\delta_{n,m}+$$

$$+e^{-nx}\sum_{k=0}^{\infty}(-m)\left(\frac{k}{n} - x\right)^{m-1}\frac{(nx)^k}{k!}+$$

$$+e^{-nx}\sum_{k=1}^{\infty}\left(\frac{k}{n} - x\right)^m\frac{(nx)^{k-1}}{(k-1)!}n =$$

$$= -n\delta_{n,m} - m\delta_{n,m-1} + \frac{n}{x}(\delta_{n,m+1} + x\delta_{n,m}).$$

Thus

$$\delta_{n,m+1} = \frac{x}{n}(\delta'_{n,m} + m\delta_{n,m-1}),$$
$$m_{n,m+1} = x(m'_{n,m} + nm\, m_{n,m-1}). \quad (2)$$

Science $\delta_{n,0} = 1$, $\delta_{n,1} = 0$, using (1) or (2) we easily find (see M. Frenţiu [5])

$$\delta_{n,2} = \frac{x}{n},\ \delta_{n,3} = \frac{x}{n^2},\ \delta_{n,4} = \frac{3x^2}{n^2} + \frac{x}{n^3},$$
$$\delta_{n,5} = \frac{10x^2}{n^3} + \frac{x}{n^4}, \quad (3)$$
$$\delta_{n,6} = \frac{15x^3}{n^3} + \frac{25x^2}{n^4} + \frac{x}{n^5}, \ldots.$$

Let us assume that for $k < m$,

$$\delta_{n,k} = \mathcal{O}\left(1/n^{]\frac{k}{2}[}\right)$$

where $]t[$ denotes the smallest integer not less that t. By (1) we have

$$\delta_{n,m} = \frac{x}{n^{m-1}}\sum_{k=0}^{m-2}\binom{m-1}{k}n^k\mathcal{O}\left(\frac{1}{n^{]\frac{k}{2}[}}\right) =$$

$$= \mathcal{O}\left(\frac{1}{n^{]\frac{m}{2}[}}\right).$$

Hence by (3) we find that

$$\delta_{n,m} = \mathcal{O}\left(\frac{1}{n^{]\frac{m}{2}[}}\right), \quad m = 2, 3, ... \tag{4}$$

Let $f(t)$ be a function bounded an all segments of non-negative real axis such that $f^{(2k)}(t)$ exists at $t = x$ and that $f(t)$ does not grow more rapidly that some power of t as $t \to \infty$. In fact using of S. Eisenberg and B. Wood [4] we may take f to be of exponential type α for some $\alpha > 0$. It follows therefore that (see [14])

$$(S_n f)(x) = f(x) + \sum_{j=1}^{2k} \frac{f^{(j)}(x)}{j!}\delta_{n,j}(x) + \frac{\varepsilon_n}{n^j}, \tag{5}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. From (3) we have

$$m_{n,0} = 1, \quad m_{n,1} = 0, \quad m_{n,2} = nx, \quad m_{n,3} = nx,$$

$$m_{n,4} = 3n^2x^2 + nx, \quad m_{n,5} = 10x^2n^2 + nx,$$

$$m_{n,6} = 15n^3x^3 + 25n^nx^2 + nx, \ldots;$$

and in general we can write $m_{n,r}$ as a polynomial in n, of the form similar to the one of Bernstein polynomials

$$m_{n,r}(x) = \psi_{r,r'}(x)n^{r'} + \psi_{r,r'-1}(x)n^{r'-1} + \\ + \psi_{r,r-'-2}(x)n^{r'-2} + \ldots + \psi_{r,1}(x)n, \tag{6}$$

which is of degree $r' = \left[\frac{1}{2}r\right]$, where $[t]$ denotes the largest integer not greater that t, with $\psi_{n,r'-i}$ being polynomials in x, independent of n.

2 The linear combination of Favard - Szász operators

We define the same combination for these operators as P.L. Butzer used for $(B_n f)(x)$. Thus the combinations $(S_n^{[2k]} f)(x)$ of $(S_n f)(x)$ are defined inductively as follows:

$$S_n^{[0]} \equiv S_n,$$

$$(2^k - 1)S_n^{[2k]} \equiv 2^k S_{2n}^{[2k-2]} - S_n^{[2k-2]}, \tag{7}$$

$$S_n^{[2k]} e_0 = e_0.$$

Then we have

$$S_n^{[2k]} = \alpha_k S_{2^k n} + \alpha_{k-1} S_{2^{k-1} n} + ... + \alpha_0 S_n, \tag{8}$$

where α_i are real constants depending on k only such that

$$\alpha_k + \alpha_{k-1} + ... + \alpha_0 = 1. \tag{9}$$

Let us note that only those values of f that are needed in computing $S_2 k_n$ are utilized in constructing $S_n^{[2k]}$.

Let us define the quantities $\xi_{n,r}^{[2k]}(x)$, $r = 1, 2, 3, \ldots; k = 0, 1, 2, \ldots; n = 1, 2, \ldots$, by

$$\xi_{n,r}^{[0]} \equiv \delta_{n,r}, \tag{10}$$

$$(2^k - 1)\xi_{n,r}^{[2k]} = 2^k \xi_{2n,r}^{[2k-2]} - \xi_{n,r}^{[2k-2]}, \quad k = 1, 2, \ldots.$$

As in the case of Bernstein polynomials we have

Lemma 1 *If $f^{(2k+2s)}(x)$ exists at the point x, then*

$$S_n^{[2k]}(x) = f(x) + \sum_{r=1}^{2(k+s)} \frac{f^{(r)}(x)}{r!}\xi_{n,r}^{[2k]}(x) + \frac{\varepsilon_n}{n^{k+s}}, \tag{11}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Proof. It is the same as that for Bernstein Polynomials. Assume (11) holds; then if $f^{(2k+2s+2)}(x)$ exists, we show that (11) holds with k replaced by $k+1$ and since (11) is true for $k = 0$ by (5), the proof will follow by induction. We have

$$S_n^{[2k]}(x) = r(x) + \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\xi_{n,r}^{[2k]}(x) + \frac{\varepsilon_n}{n^{k+s+1}},$$

replacing s by $s+1$ in (11). By (7) and (10) we have

$$(2^{k+1} - 1)\left[S_n^{[2k+2]}(x) - f(x)\right] =$$

$$= 2^{k+1}\left[S_{2n}^{[2k]} - f\right] - \left[S_n^{[2k]} - f\right] =$$

$$= 2^{k+1}\sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\xi_{2n,r}^{[2k]}(x) -$$

$$- \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\xi_{n,r}^{[2k]}(x) + \frac{\varepsilon_n}{n^{k+s+1}},$$

which proves the lemma.

Now we state our main theorem giving the approximation for $2k$ - times differentiable function by $S_n^{[2k]}(x)$. The proof is the same as for Bernstein polynomials.

Theorem 2 *If $f^{(2k)}(x)$ exists at the point x, then*

$$\left|S_n^{[2k-2]}(x) - f(x)\right| = \mathcal{O}(n^{-k}) \tag{12}$$

and

$$\left|S_n^{[2k]}(x) - f(x)\right| = o(n^{-k}), \tag{13}$$

as $n \to \infty$, $k = 1, 2, \ldots$.

Proof. By lemma 1

$$S_n^{[2k]} - f = \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!}\,\xi_{n,r}^{[2k]} + \frac{\varepsilon_n}{n^k},$$

$\varepsilon_n \to 0$, as $n \to \infty$. So, if we show that

$$\sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!}\xi_{n,k}^{[2k]} = \mathcal{O}(n^{-k-1}), \tag{14}$$

then (13) will follow. First we prove

Lemma 3 *With $\xi_{n,r}^{[2k]}$ defined by (10)*

$$\xi_{n,r}^{[2k]}(x) = 0 \text{ for } 1 \le r \le k+1, \tag{15}$$

$$\xi_{n,r}^{[2k]}(x) = \mathcal{O}(n^{-k-1}) \text{ for } r = 1, 2, 3, \ldots \tag{16}$$

To prove it, by (6) we have

$$\xi_{n,r}^{[0]}(x) = \psi_{r,r'}(x)\bar{n}^{(r-r')} + \tag{17}$$

$$+\psi_{r,r'-1}(x)\bar{n}^{(r-r'+1)} + \ldots + \psi_{r,1}(x)n^{-(r-1)}.$$

The difference operator connecting $\xi^{[2k]}$ with $\xi^{[2k-2]}$ transform n^{-s} to $(2^{k-s}-1)n^{-s}$ which is zero if $k = s$ exactly as in the case of Bernstein polynomials. Thus, operating on the right-hand side of (17) with difference operators for $s = 1, 2, 3, \ldots, k$ and omitting vanishing terms we have

$$\xi_{n,r}^{[2k]}(x) = \psi_{k+1}(x)n^{-(k+1)} + \ldots$$
$$\ldots + \psi_{r-1}(x)n^{-(r-1)}, \tag{18}$$

where the $\psi_i(x)$ are polynomials in x independent of n. This proves (16). For $k+1 > r-1$, all terms vanish and (15) follows, proving lemma.

Thus (13) and (17) follows by lemma 1 and (16), and the proof of the theorem is complete.

In particular for $k = 3$ in the theorem the explicit formulae are

$$\lim_{n\to\infty} n^3 \left[S_n^{[4]}(x) - f(x)\right] = \lim_{n\to\infty} n^3 \left[\frac{8}{3}S_{4n}(f;x) - \right.$$

$$\left. -2S_{2n}(f;x) + \frac{1}{3}S_n(f;x) - f(x)\right] = \tag{19}$$

$$= \frac{1}{8}x\frac{f^{(4)}(x)}{4!} + \frac{5}{4}x^2\frac{f^{(5)}(x)}{5!} + \frac{15}{8}x^3\frac{f^{(6)}(x)}{6!}$$

and

$$\lim_{n\to\infty} n^3 \left[S_n^{[6]}(x) - f(x)\right] =$$

$$\lim_{n\to\infty} n^3 \left[\frac{64}{21}S_{8n}(f;x) - \frac{56}{21}S_{4n}(f;x) + \right. \tag{20}$$

$$\left. + \frac{14}{21}S_{2n}(f;x) - \frac{1}{21}S_n(f;x) - f(x)\right] = 0.$$

3 Approximation of function in $C^{2k}[0, a]$

Let

$$\mathcal{Y} = \{f : [0,\infty) \to \mathbb{R}, |f(x)| \le A(f)e^{Bx},$$

$$, A(f) > 0, B > 0\}$$

and $f \in C^{2k}[0, a]$, $a > 0$.

Theorem 4 *Let $f \in \mathcal{Y} \cap C^{2k}[0, a]$, $a > 0$. Then, we have*

$$\left|(S_n^{[2k]}f)(x) - f(x)\right| \le$$

$$\le \max\left\{\frac{C}{n^k}\omega\left(f^{(2k)}; \frac{1}{\sqrt{n}}\right), \frac{C'}{n^{k+1}}\right\}, \quad x \in [0, a]$$

where $C = C(k)$ and $C' = C'(k; f)$.

Proof: With can write

$$f(t) - f(x) = \sum_{i=1}^{2k}(t-x)^i\frac{f^{(i)}(x)}{i!} +$$

$$+\frac{(t-x)^{2k}}{(2k)!}[f^{(2k)}(\eta) - f^{(2k)}(x)]\lambda(t) + (t-x)^{2m}h(t,x)$$

with $m > k$, for all $t \ge 0$, with $x \in [0, a]$ and η lying between t and x. Here λ is the characteristic function of $[0, a]$ and h is bounded by a positive constant M.

$$S_n^{[2k]}f - f = \sum_{j=0}^{k}\{\alpha_j[S_2 j_n - f]\} =$$

$$\sum_{j=0}^{k}\left\{\alpha_j\sum_{\nu=0}^{\infty}[f(2^{-j}\frac{\nu}{n}) - f(x)]t_{\nu,2^j n}(x)\right\} =$$

$$= \sum_{j=0}^{k} \alpha_j \sum_{\nu=0}^{\infty} \sum_{i=1}^{2k} (2^{-j}\frac{\nu}{n} - x)^i \frac{f^{(i)}(x)}{i!} t_{\nu,2^j n}(x) +$$

$$+ \sum_{j=0}^{k} \alpha_j \sum_{\nu=0}^{\infty} \frac{(2^{-j}\frac{\nu}{n} - x)^{2k}}{(2k)!} \left(f^{(2k)(\xi_n)} - \right.$$

$$\left. - f^{(2k)}_{(x)} \right) t_{\nu,2^j n}(x) \lambda(2^{-j}\frac{\nu}{n}) +$$

$$+ \sum_{j=0}^{k} \alpha_j \sum_{\nu=0}^{\infty} (2^{-j}\frac{\nu}{n} - x)^{2m} h(2^{-j}\frac{\nu}{n}, x) t_{\nu.2^j n}(x) =$$

$$= \sum_1 + \sum_2 + \sum_3,$$

where $\xi_j = \xi_j(\nu)$ is between x and $2^{-j}\frac{\nu}{n}$, $0 \le j \le k$. Now

$$\sum_{\nu=0}^{\infty} \sum_{i=1}^{2k} (2^{-j}\frac{\nu}{n} - x)^i \frac{f^{(i)}(x)}{i!} t_{\nu,2jn}(x) =$$

$$= \sum_{i=1}^{2k} \sum_{\nu=0}^{\infty} (2^{-j}\frac{\nu}{n} - x)^i t_{\nu,2^j n}(x) \frac{f^{(i)}(x)}{i!} =$$

$$= \sum_{i=1}^{2k} \xi^{[0]}_{2^j n,i}(x) \frac{f^{(i)}(x)}{i!}$$

Therefore

$$\sum_1 = \sum_{i=1}^{2k} \sum_{j=0}^{k} \alpha_j \xi^{[0]}_{2^j n,i}(x) \frac{f^{(i)}(x)}{i!} =$$

$$= \sum_{i=1}^{2k} \xi^{[2k]}_{n,i}(x) \frac{f^{(i)}(x)}{i!}.$$

Then from lemma 3 we have $|\sum_1| \le C_1 n^{-k-1}$.

To evaluate $\sum_2$ we proceed as follows:

$$\sum_{\nu=0}^{\infty} \frac{(2^{-j}\frac{\nu}{n} - x)^{2k}}{(2k)!} \left| f^{(2k)}(\xi_j) - \right.$$

$$\left. - f^{(2k)}_{(x)} \right| \lambda(2^{-j}\frac{\nu}{n}) t_{\nu,2^j n}(x) \le$$

$$\le \frac{\omega\left(f^{(2k)};\delta\right)}{(2k)!} \left\{ \sum_{\nu=0}^{\infty} (2^{-j}\frac{\nu}{n} - x)^{2k} t_{\nu,2^j n}(x) + \right.$$

$$\left. + \frac{1}{\delta} \sum_{\nu=0}^{\infty} |2^{-j}\frac{\nu}{n} - x|^{2k+1} t_{\nu,2^j n}(x) \right\}.$$

This expression does not exceed

$$\frac{\omega\left(f^{(2k)};\delta\right)}{(2k)!} \left\{ \frac{A_k}{(2^j n)^k} + \frac{A'_k}{\delta(2^j n)^{k+\frac{1}{2}}} \right\}.$$

Then

$$\left|\sum_2\right| \le \frac{\omega\left(f^{(2k)};\delta\right)}{(2k)!} \sum_{j=0}^{k} |\alpha_j| \left(\frac{A_k}{(2^j n)^k} + \frac{A'_k}{(2^j n)^{k+\frac{1}{2}}} \right),$$

with $\delta = n^{-\frac{1}{2}}$, we have

$$\left|\sum_2\right| \le \frac{C_2}{n^k} \omega\left(f^{(2k)}; n^{-\frac{1}{2}}\right).$$

We have

$$\left|\sum_3\right| \le M \sum_{j=0}^{k} |\alpha_j| \sum_{\nu=0}^{\infty} (2^{-j}\frac{\nu}{n} - x)^{2m} t_{\nu,2^j n}(x) \le$$

$$\le M \sum_{j=0}^{k} |\alpha_j| \frac{A_m}{(2^j n)^m} \le \frac{C_3}{n^{k+1}}.$$

The theorem follows from these estimates.

Corollary 5 *Let $f \in \mathcal{Y} \cap Lip_\alpha[0,a]$, $a > 0$. Then*

$$\left|(S^{[2k]}_n f)(x) - f(x)\right| \le M \frac{1}{n^k \sqrt{n^\alpha}}, \quad x \in [0,a]$$

where M is a constant independent of x.

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