A Solution to the Optimal Problem with Quadratic Criterion for Liniar Time Variant Systems

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Abstract: The paper establishes a non-variational procedure in order to obtain the solution to the optimal control problem. The optimal control refers to a quadratic criterion with finite final time, regarding a time-variant linear system. The proposed solution is more convenient for implementation by comparison with the classical solution. The indicated optimal controller is advantageous especially in the time-invariant case.

Keywords: optimal control, linear quadratic, time variant system

1 Introduction

One of the frequently meet optimizations problem is the linear quadratic (LQ) problem. The classical solution of this problem has some disadvantages as: the optimal controller is time variant even in the case of an invariant linear quadratic problem; also, the matriceal differential Riccati equation must be solved in inverse time [1], [2].

The paper proposes another solution, which has implementation advantages. An analytical solution for matriceal differential Riccati equation and some methods to solve the optimal control problem are indicated.

The solution to an optimal control problem can be obtained in two main connected ways: the dynamic programming based on Bellman optimum principle and the minimum principle. The last case is used in the form of the Pontriaghin principle for constrained problems, or in a simpler form for unconstrained problems – when Hamilton or Euler Lagrange equations are used [3], [4], [5], [6].

Supplementary, certain optimal control problems can be solved using non-variational methods. In some cases, the criterion can be written in a form containing a constant term and one depending on the control vector. The minimization of the last mentioned term leads to the problem solution. This solution is possible if we have a quadratic criterion.

A linear time-variant system is considered and is adopted in the present paper. Moreover, two new analytical forms for solution to Riccati differential matriceal equation are obtained.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x^0$$
$$x(t) \in \Re^n, u(t) \in \Re^m \tag{1}$$

The linear quadratic (LQ) optimal control problem refers to the system (1) and the quadratic criterion

$$I = \frac{1}{2}x^T(t_f)Sx(t_f) + \\ + \frac{1}{2}\int_{t_0}^{t_f}\left[x^T(t)Q(t)x(t) + u^T(t)P(t)u(t)\right]dt \tag{2}$$

The problem referring to the system (1) and the criterion (2) has the solution [1], [2]

$$u^*(t) = -P^{-1}(t)B^T(t)\tilde{R}(t)x(t) \tag{3}$$

where $\tilde{R}(t)$ is a symmetrical *n*x*n* matrix, solution to the Riccati matriceal differential equation

$$\dot{\tilde{R}}(t) = \tilde{R}(t)N(t)\tilde{R}(t) - \tilde{R}(t)A(t) \\ - A^T(t)\tilde{R}(t) - Q(t) \tag{4}$$

where

$$N(t) = B(t)P^{-1}(t)B^T(t) \tag{5}$$

and

$$\tilde{R}(t_f) = S \tag{6}$$

The minimum value of the criterion is

$$I^* = \frac{1}{2}x^T(t_0)\tilde{R}(t_0)x(t_0) \tag{7}$$

One can obtain the above indicated solution based on a non-variational method. This paper uses a modified form of this classical method with doubtless advantages in implementation and also presents two variants of an analytical solution to the Riccati matriceal differential equation.

2 Main results for optimal control problem

Lemma1: The control vector that minimizes the criterion (2) subject to the system (1) is

$$u^*(t) = -P^{-1}(t)B^T(t)[\bar{R}(t)x(t) + v(t)] \tag{8}$$

where the symmetric matrix $\bar{R}(t)$ is a particular solution of the equation (4), which satisfies a certain final condition

$$\bar{R}(t_f) = \bar{S}, \quad \bar{S} \geq 0, \tag{9}$$

and v(t) satisfies

$$\dot{v}(t_f) = -F^T v(t) \tag{10}$$

with

$$F(t) = A(t) - N(t)\bar{R}(t) \tag{11}$$

and

$$v(t_f) = (S - \bar{S})x(t_f) \tag{12}$$

Proof: Let us consider the scalar function

$$\pi(t) = \frac{1}{2}x^T(t)\bar{R}(t)x(t) + \frac{1}{2}x^T(t)v(t) \tag{13}$$

Tacking into account (1) , one can write

$$\dot{\pi}(t) = \frac{1}{2}(x^T A^T \bar{R}x + 2u^T B^T \bar{R}x + x^T \dot{\bar{R}}x) + \\ + \frac{1}{2}u^T B^T v + x^T \bar{R}Nv$$

(the argument t was omitted).
Adding to (2) the identity

$$\int_{t_0}^{t_f} \dot{\pi}(t)dt - \pi(t_f) + \pi(t_0) = 0,$$

yields

$$I = \frac{1}{2}x^T(t_f)(S - \bar{S})x(t_f) + \pi(t_0) + \\ + \int_{t_0}^{t_f} [\varphi(t) - \frac{1}{2}v^T(t)N(t)v(t)]dt \tag{14}$$

where

$$\varphi(t) = \frac{1}{2}x^T \bar{R}N\bar{R}x + u^T B^T \bar{R}x + \frac{1}{2}u^T Pu + \\ + u^T B^T \bar{R}x + u^T B^T v + + x^T \bar{R}Nv$$

or

$$\varphi(t) = \frac{1}{2}(u + P^{-1}B^T\bar{R}x + P^{-1}B^T v)^T P(u + P^{-1}B^T\bar{R}x + P^{-1}B^T v) \tag{15}$$

The criterion (14) depends on u(t) only through the integral of φ(t), and this is a positive definite function. Therefore, the minimum value for I is obtained when φ(t)=0. This condition is true only if u(t) satisfies (8).■

Remark 1: The classical solution (3), (4), (6) can be obtained in the same way if we adopt the scalar function

$$\pi(t) = \frac{1}{2}x^T(t)\tilde{R}(t)x(t) \tag{16}$$

Tacking into account the uniqueness of the solution to LQ problem, the function $\pi(t)$ is the same in (13) and (16).
The minimum values of the criterion is

$$I^* = \pi(t_0) + \frac{1}{2}x^T(t_f)(S - \bar{S})x(t_f) - \frac{1}{2}\int_{t_0}^{t_f} v^T(t)N(t)v(t)dt \tag{17}$$

We have to calculate v(t) in order to obtain the control vector u(t). The vector v(t) have to be expressed in terms of $x(t_0)$, which is the unique known terminal condition.
For this purpose we formulate the following

Lemma 2: The solution to the equation (10) is

$$v(t) = \Phi(t, t_0)v(t_0), \text{ or } v(t) = \Phi(t, t_f)v(t_f), \tag{18}$$

where $v(t_f)$ is given by (12), and

$$v(t_0) = \Phi(t_0, t_f)(S - \bar{S})M^{-1}(t_0, t_f)x(t_0) \tag{19}$$

In the above relations, $\Phi(t, t_f)$ is the transition matrix for $-F^T$, and

$$M(t, t_f) = \Psi(t, t_f) + \Omega_{12}(t, t_f)(S - \bar{S}), \tag{20}$$

with $\Psi(t, t_f)$ the transition matrix for F and

$$\Omega_{12}(t, t_f) = \int_t^{t_f} \Psi(t, \tau)N(\tau)\Phi(\tau, t_f)d\tau \tag{21}$$

Proof: The solution (18_2) is directly obtained from (10). Using (8), the equations (1) and (10) can be written in the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = G(t)\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \quad G(t) = \begin{bmatrix} F(t) & -N(t) \\ 0 & -F^T(t) \end{bmatrix} \tag{22}$$

The transition matrix for $G(t)\in\Re^{2nx2n}$ can be expressed as [3], [4]:

$$\Omega(t,t_f)=\begin{bmatrix}\Psi(t,t_f) & \Omega_{12}(t,t_f)\\ 0 & \Phi(t,t_f)\end{bmatrix} \qquad (23)$$

where the *n*x*n* matriceal blocks have the previous meaning. Based on the equations $\dot{\Omega}(t,t_f)=G(t)\Omega(t,t_f)$ and $\Omega(t_f,t_f)=I$ (I is the identity matrix), the relation (23) can be immediately obtained. Thus, the solution for (22) is

$$\begin{bmatrix}x(t)\\ v(t)\end{bmatrix}=\Omega(t,t_0)\begin{bmatrix}x(t_0)\\ v(t_0)\end{bmatrix} \text{ or } \begin{bmatrix}x(t)\\ v(t)\end{bmatrix}=\Omega(t,t_f)\begin{bmatrix}x(t_f)\\ v(t_f)\end{bmatrix} \qquad (24)$$

From (12), (23) and the second relation (22), yields

$$x(t)=M(t,t_f)x(t_f) \qquad (25)$$

with $M(t,t_f)$ given by (20). One can prove [5] that the matrix $M(t,t_f)$ is non-singular. This can be explained by the fact that the matrix $M(t,t_f)$ represent the transition from $x(t)$ to $x(t_f)$.

Using (12) and (24) one obtain the solution (18_1) with $v(t_0)$ given by (19).■

We are now in position to formulate supplementary remarks referring to the minimum value of the criterion.

Lemma 3: The minimum value of the criterion (2) is

$$I^*=\frac{1}{2}x^T(t_0)[\bar{R}(t_0)+\Phi(t_0,t_f)(S-\bar{S})M^{-1}(t_0,t_f)]x(t_0) \qquad (26)$$

Proof: Tacking into account (18), (21) and $\Phi^T(\tau,t_0)=\Psi^{-1}(\tau,t_0)$, the last term in (17) can be written

$$\frac{1}{2}\int_{t_0}^{t_f} v^T(t)N(t)v(t)dt=$$
$$=\frac{1}{2}v^T(t_0)\int_{t_0}^{t_f}\Phi^T(\tau,t_0)N(\tau)\Phi(\tau,t_0)d\tau v(t_0)=$$
$$=\frac{1}{2}v^T(t_0)\int_{t_0}^{t_f}\Psi(t_0,\tau)N(\tau)\Phi(\tau,t_f)d\tau\Phi^T(t_f,t_0)v(t_0)=$$
$$=\frac{1}{2}v^T(t_0)\Omega_{12}(t_f,t_0)v(t_f)$$

Replacing the previous expression in (17) and using (12), (13), (23), (24), one obtain

$$I^*=\frac{1}{2}x^T(t_0)\bar{R}(t_0)x(t_0)+$$
$$+\frac{1}{2}v^T(t_f)x(t_f)+\frac{1}{2}v^T(t_0)[x(t_0)-\Psi(t_0,t_f)x(t_f)]=$$
$$=\frac{1}{2}x^T(t_0)\bar{R}(t_0)x(t_0)+\frac{1}{2}v^T(t_0)\Phi^T(t_f,t_0)x(t_f)+$$
$$+\frac{1}{2}v^T(t_0)x(t_0)-\frac{1}{2}v^T(t_0)\Phi^T(t_f,t_0)x(t_f)$$

One immediately obtain (26) replacing $v(t_0)$ from (19).■

We can formulate now the following

Theorem 1: The solution of the optimal control problem referring to the criterion (2) and the linear system (1) is

$$u^*(t)=u_f(t)+u_c(t)\,, \qquad (27)$$

where $u_f(t)$ is the feedback component

$$u_f(t)=-P^{-1}(t)B^T(t)\bar{R}(t)x(t) \qquad (28)$$

and

$$u_c(t)=-P^{-1}(t)B^T(t)v(t) \qquad (29)$$

is a corrective component. The corrective vector $v(t)$ can be computed with (18), where $v(t_0)$ is given by (19). The minimum value of the criterion is (26).

The proof can be directly obtained from lemmas 1,2 3.

Remark 2: The above presented solution has advantages by comparison with the classical methods if one can find a particular solution of the matriceal differential Riccati equation.

3 The solution to the Riccati differential equation

An analytical formula for the solution to the Riccati differential equation can be established starting from the previous results.

Theorem 2: The solution to the Riccati differential equation (4) can be written in the forms

$$\tilde{R}(t)=\bar{R}(t)+\Phi(t,t_f)(S-\bar{S})M^{-1}(t,t_f)\,, \qquad (30)$$

or

$$\tilde{R}(t)=\bar{R}(t)+\Phi(t,t_0)WM_0^{-1}(t,t_0) \qquad (31)$$

where

$$M_0(t,t_0)=\Psi(t,t_0)+\Omega_{12}(t,t_0)[\tilde{R}(t_0)-\bar{R}(t_0)] \quad (32)$$

and the constant matrix W is given by

$$W=\Phi(t_0,t_f)(S-\bar{S})M^{-1}(t_0,t_f) \quad (33)$$

Proof: Using a similar way as in lemma 2, the corrective vector v(t) can be expressed in terms of x(t) as in (19), replacing t_0 with t. Using (13), the scalar function $\pi(t)$ can be written as

$$\pi(t)=\frac{1}{2}x^T(t)[\bar{R}(t)+$$
$$+\Phi(t,t_f)(S-\bar{S})M^{-1}(t,t_f)]x(t)$$

Comparing with (16), one obtain (30).■

In the same way, the solution can be expressed in terms of initial values

$$\tilde{R}(t)=\bar{R}(t)+\Phi(t,t_0)[\tilde{R}(t_0)-\bar{R}(t_0)]M_0^{-1}(t,t_0) \quad (34)$$

The unknown matrix $\tilde{R}(t_0)$ can be established from (30), for $t=t_0$ and, replacing in (34), one can reach (31).

Remark 3: The proposed analytical solutions for the matriceal differential Riccati equation use only nxn transition matrices, unlike the known analytical solution [1] which implies a *2n*x*2n* transition matrix.

The procedure involves the knowledge of a particular solution of this equation and the solving of a linear differential equation. This is an extension of the well known classical method for the scalar Riccati equation solution.

Let note that, the minimum values of the criterion (3) and (26) are, obviously, the same.

Remark 4: The form (31) of the solution allows the real time computation by comparison with classical methods that solve the Riccati equation in inverse time, starting from the final condition. The proposed solution can be recurrently computed. Indeed, the matrices $\Phi(t,t_0)$ and $M_0(t,t_0)$ can be recurrently computed with initial values $\Phi(t_0,t_0)=I$ and $M(t_0,t_0)=I$ (the identity matrix).

Remark 5: The time invariant linear quadratic problem, when all the matrices from (1) and (2) are constant, represents a particular important and frequently meet case. The previous lemmas and theorems can be applied replacing $\bar{R}(t)$ with the constant matrix R – solution to the Riccati algebraic equation

$$RNR-RA-A^TR-Q=0 \quad (35)$$

Obviously, we have to replace $\bar{S}$ with R, too.

4 Algorithms for optimal control

We shall refer to the time invariant problems, when the proposed methods have significant advantages, by comparison with the classical procedures. The extension to the variant time case is immediately.

In the classical methods, the optimal controller (3) is time variant even in the case of an invariant problem. This fact introduces difficulties in the controller implementation, taking into account also that the matriceal differential Riccati equation must be solved in inverse time.

Based on the previous results, one can establish a more efficient for implementation algorithms, as following:

(a) A first possibility is to use the implementation based on relation (3), but using also (30) in order to compute the matrix $\tilde{R}(t)$, with $\bar{R}(t)=R$. Note that (30) offers a simpler analytical solution by comparison with other known solutions.

(b) Another way is similar with (a) but using (31), in order to compute the solution to the matriceal differential Riccati equation. In this case, the solution is obtained in direct time and all the variant matrices from (31) can be recurrently computed.

(c) A more convenient way is based on expressions offered by the Theorem 2. Let note that, in the invariant case, the feedback component $u_f(t)$ given by (28) is a usual feedback one and is identical with the one obtained in the similar optimizations problem with infinite final time. The corrective component $u_c(t)$ given by (29) ensures the coincidence with the unique solution obtained in the finite final time problem. The vector v(t) can be recurrently computed with (18_1), with initializations (19) that depends on $x(t_0)$. This procedure has more advantages than others methods since the proposed controller is carried out only with invariant blocks.

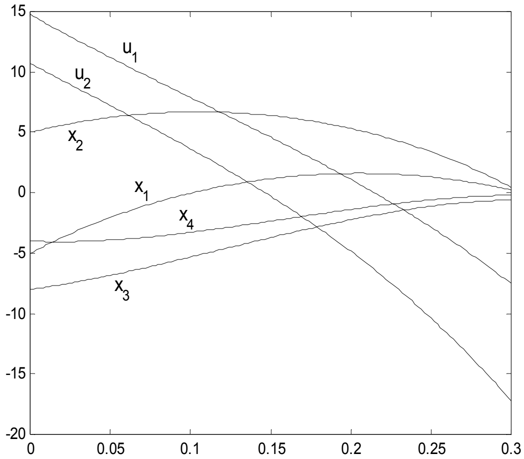
5 Simulation result

The behaviour of the optimal system was simulated for different system equations and weight matrices in the criterion. An example for 4^{th} order system with two control variables is presented in the following.

$$\dot{x}(t)=\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 2 & 4 & -1 & 0\\ 4 & 2 & 0 & -1\end{bmatrix}x(t)+\begin{bmatrix}4 & 0\\ 0 & 4\\ 0 & 0\\ 0 & 0\end{bmatrix}u(t)$$

The final time is t_f=0.3s and the matrices in (2) are chosen as follows: S=diag([10,10,10,10]), Q=diag([1,1,1,1]), p=diag([1,1]). The initial state vector is $x(0) = [-5 \quad 5 \quad -8 \quad -4]^T$.

The behaviour of the optimal control system is presented in the next figure.



time[s]

6 Conclusions

A non-variational procedure for LQ problem solution is presented.

The proposed solution is more convenient for implementation by comparison with the classical solution.

Two analytical formulas for the solution to the matriceal differential Riccati equation are also indicated. These formulas have advantages by comparison with usual methods. One of these formulas allows the direct time calculus of the solution. One of the proposed algorithms is based on the direct time solving of the Riccati equation.

It is also indicated an efficient possibility of implementation for the optimal controller, using a usual feedback and a corrective component, depending on the initial state. This optimal controller is advantageous especially in the time-invariant case.

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