Continuity Relations for Random Transients in Electrical Circuits

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Abstract: - The continuity of capacitor voltages and inductor currents, well-known from the deterministic case, cannot be directly applied if the initial conditions are random. In this paper, new continuity relations for probability densities, mean values, correlation and covariance functions of the state variables are introduced. An example illustrates the use of the new relations for a global characterization of random transients.

Deterministic transients can be regarded as particular, degenerated, random transients. On this basis, one can develop a unified analysis approach of deterministic and random transients in electrical circuits. This unified framework is certainly an advantage, first of all, in teaching activities related to transient analysis.

Key-Words: - Electrical circuit, Initial condition, Random transients, Statistical moments, Continuity relations

1 Introduction

Students find subjects related to random phenomena as difficult and vague [1]. Particularly, random transients can be analyzed using stochastic differential equations, which are also not very attractive for the average student [2], [3], [4], [5]. Therefore, at least from a teaching point of view, a simple method extending the deterministic analysis approach to random transients is a justified attempt.

It is well known that solutions of differential equations describing deterministic transients in electrical circuits are based on the continuity of capacitor voltages and inductor currents [6], [7]. Supposing transient states released by closing or opening a switch at the moment $t = 0$, the continuity of the "initial conditions", can be written as

$$v_C(+0) = v_C(-0) = v_C(0) \qquad (1)$$

for capacitor voltages, respectively,

$$i_L(+0) = i_L(-0) = i_L(0) \qquad (2)$$

for inductor currents. In these deterministic continuity relations, the notation $+0$ signifies "just after $t = 0$" and -0 means "just before" $t = 0$. The physical reasons for the above continuity relations are obvious: according to $i_C = C\frac{dv_C}{dt}$, in order for the capacitor voltage to change instantaneously, the capacitor current i_C, would have to be infinite. Similarly, in order for the inductor current to change instantaneously, the inductor voltage would have to be infinite.

This paper refers to random transients in linear electrical circuits characterized by the uncertainty of the initial condition values. Such transient states represent an intermediate step between pure deterministic transients and more complicated cases where the input signals or the system itself are random.

Section 2 presents some mathematical generalizations of the deterministic continuity relations, namely for the probability densities, mean values, covariance and correlation functions. Section 3 contains an example with analytical results and their graphical illustration. The last section is dedicated to conclusion.

2 Generalized Continuity Relations

In this section, the continuity relations (1) and (2) are generalized to include the case of random initial condition. The generalization is of pure mathematical nature and refers to probability density functions (p.d.f.), as well to first and second order statistical moments.

2.1 Continuity of probability densities

Obviously, the continuity of each possible value of a capacitor voltage assures the continuity of the p.d.f. of this voltage. This fact is expressed by the following relation:

$$p^+(v_C) = p^-(v_C) = p(v_C) \quad (3)$$

The continuity of the p.d.f. of an inductor current can be expressed by similar equalities:

$$p^+(i_L) = p^-(i_L) = p(i_L) \quad (4)$$

With the same physical justification, one can state continuity relations for mutual p.d.f. of two or more random variables.

If v_C and i_L are continuous random variables, $p(\cdot)$ in (3) and (4) are ordinary functions. However, if v_C and i_L are of discrete or mixed types, the corresponding p.d.f. contain Dirac impulses (generalized functions). In the particular case of deterministic initial conditions, the continuity relations (1) and (2) can be written as equivalent relations between probability densities:

$$\delta^+[v_C - v_C(+0)] = \delta^-[v_C - v_C(-0)] = = \delta[v_C - v_C(0)] \quad (5)$$

$$\delta^+[i_L - i_L(+0)] = \delta^-[i_L - i_L(-0)] = \delta[i_L - i_L(0)] \quad (6)$$

Thus, deterministic initial conditions can be considered as particular, degenerated cases of general random initial conditions.

2.2 Continuity of statistical moments

In electrical circuits, the capacitor voltages and inductor currents are state variables. Therefore it is suitable to consider the state-space description of a general electrical circuit (system) [8].

A continuous-time system can be described by a state equation

$$\mathbf{\dot{Z}(t) = AZ(t) + BX(t)} \quad (7)$$

and an output equation

$$\mathbf{Y(t) = CZ(t) + DX(t)}. \quad (8)$$

The general solution of this state-space system, representing the transient state vector

$$\mathbf{Z(t)} = \mathbf{Z}_{IC}\mathbf{(t)} + \mathbf{Z}_F\mathbf{(t)} \quad (9)$$

has two components. Thus,

$$\mathbf{Z}_{IC}\mathbf{(t)} = \mathbf{e^{A(t-t_0)} \cdot Z(t_0)} \quad (10)$$

represents the initial condition response, also called the zero-input solution (IC solution). The second component of the state vector,

$$\mathbf{Z}_F\mathbf{(t)} = \int_{\mathbf{t_0}}^{\mathbf{t}} \mathbf{e^{A(t-\tau)} \cdot B \cdot X}(\tau) \cdot \mathbf{d\tau} \quad (11)$$

is the forced solution, caused by the forcing input vector, $\mathbf{X}(\tau)$.

For an *n*-order system (circuit), the vector of the mean values of the state variables at moment $t = 0$, can be written in the form

$$m_{\mathbf{Z}}(0) = \|E\{Z_i(0)\}\| = \|m_i(0)\|; \quad i = 1, 2, ..., n \quad (12)$$

where $E\{\cdot\}$ and $\|\cdot\|$ are denoting mathematical expectation and the matrix notation, respectively. In addition to the initial mean values, two second order statistical moments at $t = 0$ are important characteristics of the state vector: the correlation matrix

$$R_{\mathbf{Z}}(0,0) = \|E\{Z_i(0) \cdot Z_j(0)\}\| = = \|r_{ij}(0)\|; \quad i, j = 1, 2, ..., n \quad (13)$$

and the covariance matrix

$$C_{\mathbf{Z}}(0,0) = \|E\{[Z_{Ci}(0)] \cdot [Z_{Cj}(0)]\}\| = = \|c_{ij}(0)\|; \quad i, j = 1, 2, ..., n \quad (14)$$

In (14), $Z_C(\cdot) = Z(\cdot) - E\{Z(\cdot)\}$ represents a centred component of the state vector.

The continuity of the statistical moments is a direct consequence of the continuity of the probability densities. For example, one can write

$$\int_{-\infty}^{\infty} z_i \cdot p^+(z_i) \cdot dz_i = \int_{-\infty}^{\infty} z_i \cdot p^-(z_i) \cdot dz_i = = \int_{-\infty}^{\infty} z_i \cdot p(z_i) \cdot dz_i \quad (15)$$

or [1]

$$E\{Z_i(0^+)\} = E\{Z_i(0^-)\} = = E\{Z_i(0)\}; \quad i = 1, 2, ..., n \quad (16)$$

From (16) follows the continuity of the mean values vector:

$$m_{\mathbf{Z}}(+0) = m_{\mathbf{Z}}(-0) = m_{\mathbf{Z}}(0). \quad (17)$$

Using the continuity of joint densities of any two variables from the state vector,

$$p^+(z_i, z_j) = p^-(z_i, z_j) = p(z_i, z_j) \quad i, j = 1, 2, ..., n \quad (18)$$

we conclude the equality of second order expected values

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \cdot p^+(z_i, z_j)\, dz_i dz_j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \cdot p^-(z_i, z_j)\, dz_i dz_j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \cdot p(z_i, z_j)\, dz_i dz_j. \quad (19)$$

and, finally, the continuity of the correlation matrix:

$$R_{\mathbf{Z}}(+0,+0) = R_{\mathbf{Z}}(-0,-0) = R_{\mathbf{Z}}(0,0)\,. \quad (20)$$

Using the continuity of joint p.d.f., the continuity of the covariance matrix can also be put into evidence. However, the continuity of the covariance can be proved from the continuity of the correlation and the mean functions. Actually, taking (17) and (20) into account and the following equalities for $t_1 = t_2 = 0$,

$$\begin{aligned} c_{ij}(t_1,t_2) &= M\{\lfloor Z_{Ci}(t_1)\rfloor \cdot \lfloor Z_{Cj}(t_2)\rfloor\} = \\ &M\{Z_i(t_1)\cdot Z_j(t_2)\} - \\ &- M\{Z_i(t_1)\}\cdot M\{Z_j(t_2)\} = \\ &r_{ij}(t_1,t_2) - m_i(t_1)\cdot m_j(t_2); \quad i,j = 1,2,...,n. \end{aligned} \quad (21)$$

the continuity of the correlation matrix follows:

$$C_{\mathbf{Z}}(+0,+0) = C_{\mathbf{Z}}(-0,-0) = C_{\mathbf{Z}}(0,0)\,. \quad (22)$$

The continuity relations (3), (4), (17), (20) and (22) can be used in the calculation of random transients. Deterministic transients can be regarded as degenerated random transients, where the corresponding p.d.f. are impulse (Dirac) functions.

3 An Example

In order to illustrate the utilization of the derived continuity relations, we consider the simple example of a first-order RC low-pass filter with constant input voltage: $x(t) = 1\mathrm{V}$ for $t \geq 0$. The coefficients of state equations (7) and (8) are: $A = -a$; $B = a$; $C = 1$ and $D = 0$ with $a = 1/RC$; For numerical computations the following values are considered: $R = 10k\Omega$; $C = 1\mu F$ resulting a time constant $RC = 10ms$ and $a = 100Hz$. The single state variable (the voltage on the capacitor) is also the output signal, $z(t) = y(t)$. In this case, the state-space equations can be expressed as

$$\dot{y}(t) = -a\cdot y(t) + a\cdot x(t) \quad (23)$$

with solution

$$y(t) = e^{-at}\cdot y(0) + 1 - e^{-at}; \quad t \geq 0\,. \quad (24)$$

Contrary to the usual assumption, we consider that the initial voltage on the capacitor is unknown and has a uniform p. d. f.

$$p_Y^-(y) = p_Y(y;-0) = \begin{cases} \dfrac{1}{v_2 - v_1} & v_1 \leq y \leq v_2 \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

We can now apply the continuity relation

$$p_Y^-(y) = p_Y^+(y) = p_Y(y) = \Pi(y;v_1,v_2) \quad (26)$$

Since $y(0)$ is unknown, according to (24), the output voltage is a linear transformation of the random variable $Y(0)$:

$$Y(t) = \alpha\cdot Y(0) + \beta$$

where $\alpha = e^{-at}$ and $\beta = 1 - e^{-at}$. The p.d.f. of the output variable $Y(t)$ is [1]:

$$p_Y(y;t) = \Pi(y;\alpha v_1 + \beta, \alpha v_2 + \beta) = \begin{cases} \dfrac{1}{\alpha(v_2 - v_1)} & \alpha v_1 + \beta \leq y \leq \alpha v_2 + \beta \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

This p.d.f. is represented in Fig.1 for $v_1 = -1\mathrm{V}$, $v_2 = 2\mathrm{V}$ at five different moments during the transient process: $t = 0ms$, $10ms$, $20ms$, $30ms$ and $40ms$. The transient p.d.f. shows that with increasing time the random effect of the initial condition disappears and the output voltage becomes deterministic. The uniform p.d.f. approaches a Dirac impulse, for $t \to \infty$.

$$\lim_{t\to\infty} p_Y(y,t) = \delta(y-1) \quad (28)$$

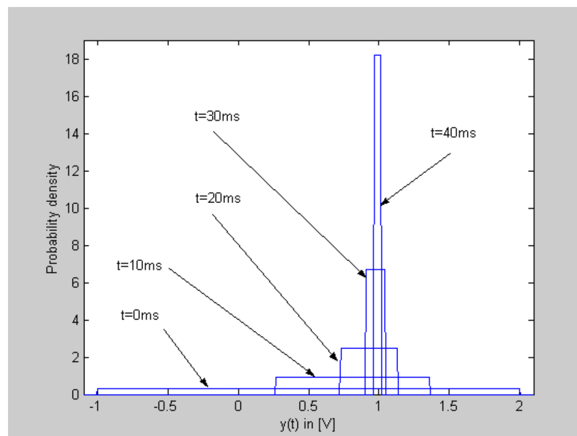

Fig.1 Probability density function of the output voltage for five distinct time values

The initial condition of the mean value is

$$m_Y(0) = \int_{v_1}^{v_2} y\frac{1}{v_2 - v_1}dy = \frac{v_1 + v_2}{2}\,. \quad (29)$$

The transient mean value has the expression

$$\begin{aligned} m_Y(t) &= E\{Y(t)\} = \\ &= e^{-at}m_Y(0) + 1 - e^{-at}. \end{aligned} \quad (30)$$

For $v_1 = -1$ and $v_2 = 2$, the mean value of the output voltages can be written as

$$m_Y(t) = 1 - 0.5\cdot e^{-at} \quad (31)$$

This particular mean value is represented in Fig.2, together with four transient output voltages corresponding to $y(0) = -0.5\mathrm{V}$, $0\mathrm{V}$, $1.5\mathrm{V}$ and $2\mathrm{V}$.

Obviously, the mean value can be regarded as the output voltage for $y(0)=0.5\text{V}$ initial condition.

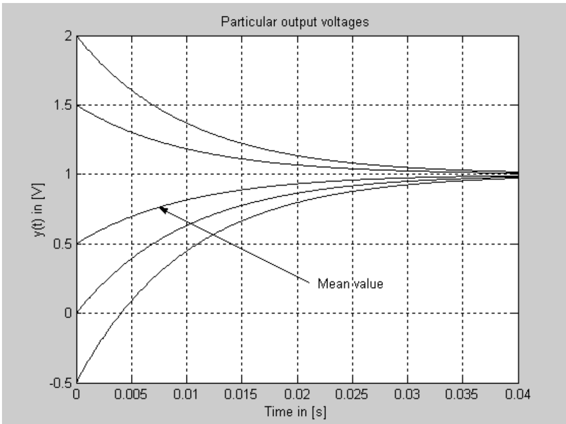

Fig.2 Particular transient voltages on the capacitor for different initial conditions

In order to determine the initial autocorrelation function we express the joint p.d.f. using the conditional p.d.f. [1]:

$$p_Y^-(y_1,y_2)=p_Y^-(y_1)\cdot p_Y^-(y_2\mid y_1)= \\ =\Pi(y_1,v_1,v_2)\cdot\delta(y_2-y_1) \quad (32)$$

It follows, for every $t_1, t_2 \le 0$,

$$R_Y(t_1,t_2)=M\{Y(t_1)Y(t_2)\}= \\ =\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1y_2\cdot\Pi(y_1,v_1,v_2)\cdot\delta(y_2-y_1)dy_1dy_2= \\ =\frac{v_1^2+v_1v_2+v_2^2}{3}=R_Y(0,0). \quad (33)$$

According to the definition of the autocorrelation function we obtain the general expression:

$$R_Y(t_1,t_2)=M\{Y(t_1)Y(t_2)\}= \\ =e^{-a(t_1+t_2)}\cdot R(0,0)+ \\ +e^{-at_1}\left(1-e^{-at_2}\right)\cdot m_Y(0)+ \\ +e^{-at_2}\left(1-e^{-at_1}\right)\cdot m_Y(0)+ \\ +\left(1-e^{-at_1}\right)\cdot\left(1-e^{-at_2}\right) \quad (34)$$

For the particular values $v_1=-1\text{V}$, $v_2=2\text{V}$, the transient autocorrelation function equals

$$R_Y(t_1,t_2)=e^{-a(t_1+t_2)}- \\ -e^{-at_1}-e^{-at_2}+1 \quad (35)$$

This last expression is represented in the Fig.3, for the domain $0\le t_1,t_2\le 40ms$. Obviously, with increasing t_1 and t_2, the autocorrelation approaches a constant value equal to the square of the constant steady-state value of the output voltage.

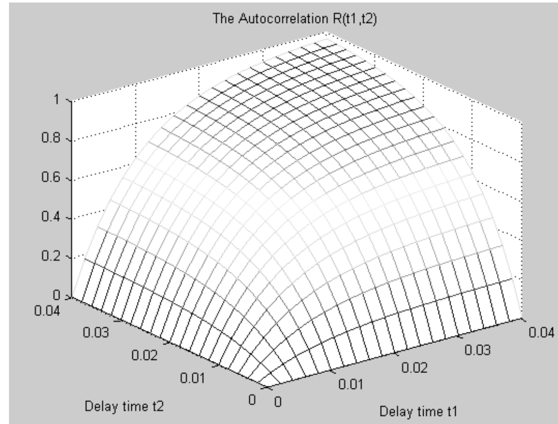

Fig.3 The autocorrelation function $R_Y(t_1,t_2);\quad 0\le t_1,t_2\le 40ms$

The initial value of the covariance function can be calculated using the centered initial p.d.f.

$$\Pi\left(y;\frac{v_1-v_2}{2},\frac{v_2-v_1}{2}\right)$$

It follows,

$$C_Y(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1y_2\cdot\Pi\left(y;\frac{v_1-v_2}{2},\frac{v_2-v_1}{2}\right)\cdot \\ \cdot\delta(y_2-y_1)dy_1dy_2= \\ =\frac{(v_2-v_1)^2}{12}=C_Y(0,0) \quad (36)$$

Finally, we obtain the transient covariance function,

$$C_Y(t_1,t_2)=C_Y(0,0)\cdot e^{-a(t_1+t_2)}. \quad (37)$$

For $v_1=-1$ and $v_2=2$, the covariance has the particular expression:

$$C_Y(t_1,t_2)=\frac{3}{4}\cdot e^{-a(t_1+t_2)} \quad (38)$$

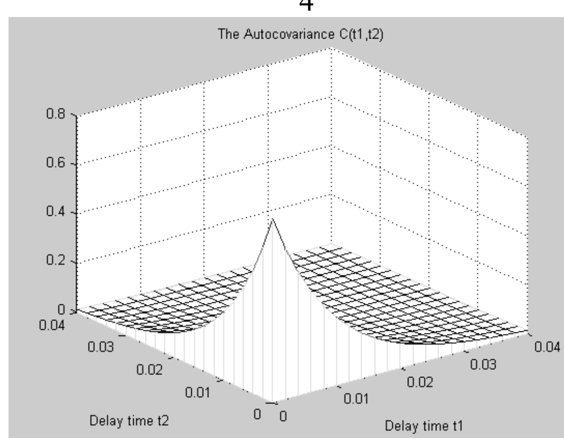

Fig.4 The covariance function $C_Y(t_1,t_2);\quad 0\le t_1,t_2\le 40ms$

This covariance function is represented in Fig.4. For increasing t_1 and t_2 the covariance approaches zero because the steady-state value of the output voltage is a constant.

One can also observe from this example, that the transient p.d.f., mean value, correlation and covariance functions offer a global description of all possible transients of the state variables or output signals in an electrical circuit. This global characterization is a valuable alternative in the case when, due to the random initial condition, one can not specify a particular, deterministic, transient process.

4 Conclusion

This paper presents continuity relations for probability densities, mean values, correlation and covariance functions of state variables in electrical circuits. The introduced relations are mathematical generalizations of the well known initial condition continuity relations from the deterministic case. A simple example illustrates the use of these new relations in a global characterization of random transients.

Deterministic transients can be regarded as particular, degenerated random transients. On this basis one can develop a unified analysis approach of deterministic and random transients in electrical circuits. A unified framework is certainly an advantage, first of all, in teaching activities related to transient analysis.

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