Algorithm for Defining the Distribution of Zeros of Random Polynomials

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Abstract: - The problem of defining the distribution of real zeros for random n-th order polynomials whose coefficients have given continuous joint probability density function considered. A new algorithm for defining the distribution of real zeros via multiple integration presented. A theorem validating the algorithm proved. Realization and compatibility of the algorithm discussed.

Key-Words: - Random polynomial , Fourier- Budan theorem, Convex polyhedron, Multiple integral, Numerical method

1 Introduction

Analysis of behavior of zeros of random polynomials has been a subject of active research for several decades motivated by various applications in statistics , spectral analysis, physics, economy and other fields (some new results on random polynomials and their applications are described in [1],[5],[8],[11]). One of the main problems in the theory of random polynomials is defining the distribution or real zeros (see [10],[11]). Computer simulation remains the only available general method for solving the problem for arbitrary continuous joint probability density function (jpdf) of polynomial coefficients even for polynomials of relatively low degrees. Unfortunately, standard simulation algorithms and software can't be effectively utilized for solving the problem, therefore developing specific methods for defining the distribution of real zeros of random polynomials seems highly desirable. The latter motivated us to develop the algorithm presented in the current article .

In order to describe the algorithm and prove its validity we introduce some notations.

Let $\vec{a} = (a_0,...,a_n)$ denote a vector in (n+1)-dimensional Euclidean space R^{n+1}.Let $F_n(\vec{a},x)$ denote the polynomial

$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, the coefficients of which are elements of $\vec{a}$.

Let $S = \{\vec{a} :| a_0 |< s_0,| a_1 |< s_1,...,| a_n |< s_n\}$, $i = \overline{0,n}$ denote a cube in R^{n+1} which defines the boundaries for the coefficients.

Let *B* denote the union of *k* nonintersecting intervals $B_i = (l_i, d_i), i = \overline{1,k}$ in *R*.

Let $\vec{a}(\omega) = (a_0(\omega), a_1(\omega),..., a_n(\omega))$ denote an (*n+1*) – dimensional random vector ,whose elements are random variables, not necessarily independent, which are coefficients of the random polynomial $F(\vec{a}(\omega), x) = a_0(\omega) + a_1(\omega)x + ... + a_n(\omega)x^n$

Let $p(\vec{a})$ denote the joint probability density function of random coefficients.

Let $Q_{n_1,n_2,...n_k}(B) \in S$ denote the set of all vectors $\vec{a}$ in S for which the corresponding polynomials $F(\vec{a},x)$ have n_1 zeros in B_1, n_2 zeros in B_2,…, n_k zeros in B_k and

$F(\vec{a},l_i) \neq 0$, $F(\vec{a},r_i) \neq 0, i = \overline{1,k}$ is satisfied.

Obviously , $n_1 + n_2 + ... + n_k \leq n$ must be satisfied.

Let $P_{n_1,n_2,...n_k}(B)$ denote the probability that $F(\vec{a}(\omega),x)$ has n_1 zeros in B_1 , n_2 zeros in B_2, …, n_k zeros in B_k .

Let $Q^*_{n_1,n_2,...n_k}(B)$ denote the set of vectors $\vec{a}$ in *S* for which $l_i, r_i, i = \overline{1,k}$ are not zeros of $F(\vec{a},x)$, the number of zeros of $f(\vec{a},x)$ belonging to B_1 equals n_1 or equals n_1 plus an even integer, the number of zeros that belong to B_2 equals n_2 or equals n_2 plus an even integer , etc.

Let $P^*_{n_1,n_2,...n_k}(B)$ denote the probability that $F(\vec{a}(\omega),x)$ has n_1 zeros in B_1 or n_1 plus an even number of zeros in B_1, n_2 zeros or n_2 plus an even number of zero in B_2, etc.

Since the Euclidean measure of the set of vectors $\vec{a}$ which have one of the numbers $l_i, r_i, i=\overline{1,k}$ as their zero, obviously equals zero , we have

$$(1)\ P_{n_1,n_2,...n_k}(B) = \int_{Q_{n_1,...,n_k}(B)} p(\vec{a})d\vec{a}$$

$$(2)\ P^*_{n_1,n_2,...n_k}(B) = \int_{Q^*_{n_1,...,n_k}(B)} p(\vec{a})d\vec{a}$$

In order to calculate the integral (2), we need to examine $Q^*_{n_1,n_2,...n_k}(B)$.

2 Statement and Proof of the Theorem

We formulate the theorem which validates the algorithm presented in section 3.

Theorem 1 *$Q^*_{n_1,n_2,...n_k}(B)$ belongs to a union of a finite number (which is less than $2^{(2n+2)k}$) of nonintersecting convex polyhedrons in S.*

Proof: The proof of the Theorem 1 is based on the well-known Fourier-Budan theorem (see [4]).

Fourier- Budan theorem *For any real polynomial p(x) of degree n and any real α and β such as α<β, ρ(α)≠0 and ρ(β)≠0 the number of zeros in the interval [α, β] (each zero counted with proper multiplicity) equals υ(α)-υ(β) minus an even nonnegative integer, where υ(x) denotes the number of sign changes in the sequence $\{p(x), p'(x),..., p^n(x)\}$.*

Let's first examine the set $Q_{n_1}(B_1)$ in *S* consisting of all vectors $\vec{a}$ for which the corresponding polynomial has exactly n_1 zeros in B_1.

Consider the set $SMP = \{SMP_i(n_1)\}_{i=1}^{N_1}$ of all sequences of pluses and minuses which consist of 2 subsequences $SMP_i^l(n_1)$ and $SMP_i^r(n_1)$ for which $\upsilon(SMP_i^l(n_1)) - \upsilon(SMP_i^r(n_1))$ *equals* n_1 or equals n_1 plus an even integer .The number N_1 of all distinct sequences belonging to *SMP* can be easily calculated. For each sequence $SMP_i(n_1)$ from *SMP* we have a corresponding system of linear inequalities $SYS(SMP_i(n_1), B_1)$. The first inequality that belongs to $SYS(SMP_i(n_1), B_1)$ has the form $F(\vec{a},l_1) = a_0 l_l^n + ... + a_n > 0$ if the first element of $SMP_i(n_1)$ is "+" or has the form $F(\vec{a},l_1) < 0$ if the first element of $SMP_i(n_1)$ is "-" .The second inequality in $SYS(SMP_i(n_1), B_1)$ has the form $F^1(\vec{a},l_1) = na_0 l_1^{n-1} + (n+1)a_1 l_1^{n-2} + ... + a_{n-1} > 0$ if the second element of $SMP_i(n_1)$ is "+" or the form $F^1(\vec{a},l_1) < 0$ if the second element of $SMP_i(n_1)$ is "-".The following inequalities up to *n*-th are defined analogously.

The *(n+1)* –th inequality in $SYS(SMP_i(n_1), B_1)$ is $F^n(\vec{a},l_1) = n!a_0 > 0$ or $F^n(\vec{a},l_1) < 0$ according to the corresponding element of $SMP_i(n_1)$. The inequalities with ordinary numbers from *n+2* to *2n+2* are defined analogously to the previous ones, we just substitute *r* instead of *l* in each of the previous inequalities. The double inequalities $-s_j < a_j < s_j$, $j=\overline{0,n}$ are also included in $SYS(SMP_i(n_1), B_1)$. In view of Fourier-Budan theorem a polynomial $F(\vec{a},x)$ has n_1 or n_1 plus an even integer zeros in B_1 if its coefficients satisfy one of the systems of inequalities $SYS(SMP_i(n_1), B_1), i=\overline{1,N_1}$. The system of inequalities $SYS(SMP_i(n_1), B_1)$ defines a convex polyhedron in *S*. Let $PH_i(n_1, B_1)$ denote the set of points $\vec{a} \in R^{n+1}$ that satisfy $SYS(SMP_i(n_1), B_1)$. We have

$$Q^*_{n_1}(B_1) \subset \bigcup_{i=1}^{N_1} PH_i(n_1, B_1)$$ .Analogously

$$Q^*_{n_2}(B_2) \subset \bigcup_{i=1}^{N_2} PH_i(n_2, B_2),...,$$

$$Q^*_{n_k}(B_k) \subset \bigcup_{i=1}^{N_k} PH_i(n_k, B_k).$$

$Q^*_{n_1,n_2,\ldots n_k}(B) \subset \bigcap_{j=1}^{k}\bigcup_{i=1}^{N_j} PH_i(n_i, B_i)$, from which follows that it belongs to the union of a finite number of nonintersecting convex polyhedrons since the intersection of two or more convex polyhedrons that belong to S is a convex polyhedron.

Corollary 1 (2) *equals the sum of a finite number of multiple integrals, each of which is calculated over a convex polyhedron defined by a system of linear inequalities , the polyhedrons have no intersections, and the integrand is*

$p(\vec{a}) * I(Q^*_{n_1,n_2,\ldots n_k}(B))$ *,where*

$$I(Q^*_{n_1,n_2,\ldots n_k}(B)) = \begin{cases} 1, \vec{a} \in Q^*_{n_1,n_2,\ldots n_k}(B) \\ 0 \text{ , othervise} \end{cases}$$

Corollary 2 *In order to state Corollary 2 we just need to substitute (1) instead of (2) in the statement of Corollary 1.*

We do not find it necessary to give the proof, but just note that, obviously, the probability that $F(\vec{a}(\omega), x)$ has exactly n_1 zeros in B_1 , n_2 or more zeros in B_2 , n_3 or more zeros in B_3 ,etc. ,equals

(3) $P^*_{n_1,n_2,\ldots n_k}(B) - P^*_{n_1+2,n_2,\ldots n_k}(B)$

and the probability that $F(\vec{a}(\omega, x))$ has exactly n_1 zeros in B_1 , exactly n_2 zeros in B_2, n_3 or more zeros in B_3, n_4 or more zeros in B_4,etc., can be expressed as

(4) $P^*_{n_1,n_2,\ldots n_k}(B) - P^*_{n_1,n_2+2,\ldots n_k}(B) - (P^*_{n_1+2,n_2,\ldots n_k}(B) - P^*_{n_1+2,n_2+2,\ldots n_k}(B))$

which shows how the formula which gives an explicit expression for (1) as a sum of multiple integrals over convex polyhedrons can be obtained.

3. Algorithm Description

Corollary 2 enables one to present the following algorithm for defining the probability that a random polynomial $F(\vec{a}(\omega), x)$ has exactly n_1 zeros in B_1, n_2 zeros in B_2,..., n_k zeros in B_k provided that the given joint probability density function of coefficients is continuous (joint probability density function may take non-zero values in some bounded domain D in R^{n+1} or be positive for any $\vec{a} \in R^{n+1}$ which is the case for Gaussian random variables).

Step 1: Define the boundaries for random variables $a_0(\omega), a_1(\omega), \ldots, a_n(\omega)$.If the probability density function takes non-zero values in some bounded domain D in R^{n+1}, the minimal possible (n+1)-dimensional cube of the form S has to be defined which includes D . If the probability density function is positive for any $\vec{a} \in R^{n+1}$, the minimal cube S should be defined which satisfies $\int_S p(\vec{a}) d\vec{a} > 1 - \varepsilon$, where ε denotes the accuracy required in calculating (1) .

Step 2: Define all the systems of inequalities (polyhedrons), the integration over which is necessary in order to calculate the desired probability.

Step 3: Calculate the multiple integrals over the defined polyhedrons.

Step 4: Summarize the calculated integrals.

4. Algorithm Realization

The main advantage of the presented algorithm is a significant restriction of the integration domain .

In many cases the volume of $Q^*_{n_1,n_2,\ldots,n_k}(B)$ is 10^{n+1} times smaller than the volume of S.

In order to realize the presented algorithm , the software for multiple integration over convex polyhedrons (step 3) must be combined with software for realization of step 2.Recently developed routines for multiple integration over complicated domains are described in [7] and [12], and a routine designed for integration over convex polyhedrons is presented in [9]. All these routines can be utilized in realization of the presented algorithm with nonsignificant modifications.

5. Conclusive Remarks

Remark 1 The number of integrals which need to be calculated according to the presented algorithm grows rapidly with the growth of n ,but for relatively small values of n the calculation of probabilities (1) via computer routines implementing the algorithm appear to be effective and compatible.

Remark 2 Developing the software routines implementing the presented method , defining the maximum value of *n* for which the algorithm is realizable, estimation of its effectiveness and comparison with alternative methods are the main fields of planned future research.

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