An interpolation problem in generalized degree polynomial spaces

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Abstract: The aim of this paper is to study many interpolation problems in the space of polynomials of w-degree n. In order to do this, some new results concerning the polynomial spaces of w-degree n are given. We consider only the case of functions in two variables, but all the results obtained can be easily extended to many variables. We found a set of conditions for which, $\Pi_{n,w}$, the space of polynomials of w-degree is an interpolation space. More details are obtained for the weight $w = (1, w_2)$.

Key–Words: Generalize degree, Multivariate polynomial interpolation, w- homogeneous polynomial spaces

1 Introduction

The processes modeling, often requires the approximation of some unknown multivariate functions. In many cases we know only some information about these functions. Usually these information is represented by means of the values of some linear functionals applied to the unknown function. When it is necessary to approximate the functions, matching the known information, the interpolation is used. By computational reasons, polynomial interpolation is preferred an more, one looks for minimal interpolation spaces.

Various interpolation scheme were studied, connected to various modeling problems:

- Lagrange interpolation, with the set of conditions

  $$\Lambda = \{\delta_\theta(f) = f(\theta) | \theta \in \Theta\}.$$

  (see [1], [2], [3], [4], [6], [8], etc).

- Hermite interpolation, defined in many ways and involving certain derivatives of the unknown function. A general way of describing the Hermite conditions is given in [5]

  $$\Lambda = \{\lambda_{q,\theta} \, | \lambda_{q,\theta}(p) = (q(D)p)\,(\theta)\,\},$$

  $q \in \mathcal{P}_\theta;\ \theta \in \Theta;\ \mathcal{P}_\theta \subset \Pi.$
  Other definition can be found in [9] and uses chains of derivatives, organized in a tree.

- An interpolation scheme, for a set of general functionals, Λ, is given in [3]. A minimal interpolation space for Λ is

  $$H_\Lambda\!\downarrow = span\{g\!\downarrow; g \in H_\Lambda\} \qquad (1)$$

  with

  $$H_\Lambda = span\{\lambda^\nu;\ \lambda \in \Lambda\} \qquad (2)$$

  and λ^ν being the generating function of the functional λ.

- We studied in [11] an interpolation scheme which use as set of conditions the values of the homogeneous parts of the function, that is, we used the conditions

  $$\delta_{j,\theta_k} : \mathcal{A}_0 \to R;$$

  $$(\delta_{j,\theta_k})\,(f) = f^{[j]}(\theta_k),\ \theta_k \in \Theta \subset R^d,$$

  with $f^{[j]}$, the homogeneous part of degree j from Taylor series of f.

There are many others particular schemes investigated and applied in practical studies.

The aim of this article is to study a particular interpolation scheme, using a generalized degree for the polynomials it works with.

In 1994, T. Sauer proposed, in [10], a generalization of the degree, using a weight $w \in N^d$.

Definition 1 *([10]) The w-degree of the monomial x^α is*

$$\delta_w(x^\alpha) = w \cdot \alpha = \sum_{i=1}^{d} w_i \cdot \alpha_i \qquad (3)$$

$\forall\, \alpha \in N^d,\ w = (w_1, \ldots, w_d) \in N^d,\ x \in R^d.$

We will denote by $\Pi_{n,w}$ the vector space of all polynomials of w- degree less than or equal to n and we will denote by $\Pi^0_{n,w}$ the vector space of all homogeneous polynomials of total w- degree exactly n:

$$\Pi_{n,w} = \left\{ \sum_{w\cdot\alpha\leq n} c_\alpha x^\alpha \mid c_\alpha \in R,\ \alpha \in N^d \right\} \quad (4)$$

$$\Pi^0_{n,w} = \left\{ \sum_{w\cdot\alpha = n} c_\alpha x^\alpha \mid c_\alpha \in R,\ \alpha \in N^d \right\} \quad (5)$$

We will use the notations from [12]:

$$A^0_{n,w} = \{\alpha \in N^d,\ w\cdot\alpha = n\}, \quad (6)$$

$w \in (N^*)^d,\ n \in N$ and

$$r_w(n) = \#(A^0_{n,w}) \quad (7)$$

The polynomial homogeneous subspace of w- degree n can be rewritten as:

$$\Pi^0_{n,w} = \left\{ \sum_{\alpha\in A^0_{n,w}} c_\alpha x^\alpha \mid c_\alpha \in R,\ \alpha \in N^d \right\}$$

We observe that $r_w(n)$ is the dimension of the w-homogeneous subspace $\Pi^0_{n,w}$.

The w-degree is a H-grading in the sense of [10], be cause it satisfy the property:
$\delta_w(x^{\alpha+\beta}) = \delta_w(x^\alpha) + \delta_w(x^\beta)$
and those

$$\Pi_{n,w} = \bigoplus_{k\in M_n} \Pi^0_{k,w},$$

with $M_n = \{k \in N \mid r_w(k) > 0 \text{ and } k \leq n\}$.
We denote by

$$f^{[j]_w} = \sum_{\alpha\cdot w = j} \frac{(D^\alpha f)(0)x^\alpha}{\alpha!},$$

the w-homogeneous part of f and by $f\downarrow_w$, the w-least term of f, that is the term wit the lest w-degree in Taylor series of f.
The interpolation conditions for the problem we consider are of the following type:

$$\lambda_{j,k} = f^{[j]_w}(\theta_{j,k}), \quad (8)$$

$\theta_{j,k} \in \Theta \subset R^2, j \in \{1,\ldots,n\}$.
This type of conditions appears when the coordinates of points are one spatial coordinate and one temporal coordinate. Our problem can be easily extend for $d > 2$ coordinates. We considered the case $d = 2$, only for computational reasons.

2 Some results concerning the space of polynomials of w - degree

The dimensions of the homogeneous w- space, $r_w(n)$ and the set $A^0_{n,w}$ depend of the weight $w = (w_1, w_2) \subset Z^2_+$. In [12], we found a general expression for $r_w(n)$ and $A^0_{n,w}$, for arbitrary $w_1,\ w_2$ and implemented two variants of algorithms based on this expression. These results are given in theorem 1.

Theorem 1 *([12]) Let $w = (w_1, w_2) \in (N^*)^2$ and let consider the functions:*

$$r : \{0,\ldots,w_1-1\} \to \{0,\ldots,w_1-1\};$$
$$r(i) = (iw_2)\ mod\ w_1$$
$$\tilde{r} : \{0,\ldots,w_2-1\} \to \{0,\ldots,w_2-1\};$$
$$\tilde{r}(i') = (i'w_1)\ mod\ w_2$$

Then

1. *$r_w(0) = 1$ and $(0,0) \in A^0_{0,w}$.*
2. *If $j = cw_1$, then $r_w(j) = \left[\frac{c}{w_2}\right] + 1$, and $(\alpha_1,\alpha_2) \in A^0_{j,w}$ are given by*
$$\alpha_2 = kw_1,\ \text{with } 0 \leq k \leq \left[\frac{c}{w_2}\right];$$
$$\alpha_1 = \frac{j - w_2\alpha_2}{w_1}.$$
3. *If $j = cw_2$, then $r_w(j) = \left[\frac{c}{w_1}\right] + 1$, and $(\alpha_1,\alpha_2) \in A^0_{j,w}$ are given by*
$$\alpha_1 = kw_2,\ \text{with } 0 \leq k \leq \left[\frac{c}{w_1}\right];$$
$$\alpha_2 = \frac{j - w_1\alpha_1}{w_2}.$$
4. *For any j, $0 < j < \min(w_1,w_2)$, $r_w(j) = 0$.*
5. *If $j \not\vdots w_1$ and $j \not\vdots w_2$, with $j \geq \min(w_1,w_2)$, then $r_w(j) = \#(M_1)$ with*
$$M_1 = \left\{ \left[0, \left[\frac{j - iw_2}{w_1w_2}\right]\right] \cap N \right\},$$
if $\frac{j - iw_2}{w_1w_2} \geq 0$, where $[\cdot]$ is the integer part function, $i = r^{-1}(s)$, with $s = j\ mod\ w_1$, and $(\alpha_1,\alpha_2) \in A^0_{j,w}$ are given by
$$\alpha_1 = \frac{j - (q_1w_1 + i)w_2}{w_1},$$

with $q_1 \in M_1$ *and*

$$\alpha_2 = \frac{i - w_1\alpha_1}{w_2}.$$

6. *If* $j \not\vdots w_1$ *and* $j \not\vdots w_2$*, with* $j \geq \min(w_1, w_2)$*, then* $r_w(j) = \#(M_2)$ *with*

$$M_2 = \left\{ \left[0, \left[\frac{j - i'w_1}{w_1w_2}\right]\right] \cap N \right\},$$

if $\frac{j - i'w_1}{w_1w_2} \geq 0,$ *where* $[\cdot]$ *is the integer part function,* $i' = \tilde{r}^{-1}(p)$*, with* $p = j \ mod \ w_2$*, and* $(\alpha_1, \alpha_2) \in A^0_{j,w}$ *are given by*

$$\alpha_2 = \frac{j - (q_2w_2 + i')w_1}{w_2},$$

with $q_2 \in M_2$ *and*

$$\alpha_1 = \frac{i - w_2\alpha_2}{w_1}.$$

7. *If* $j \not\vdots w_1$ *and* $j \not\vdots w_2$*, with* $j \geq \min(w_1, w_2)$ *and* $\min\{j - iw_1, j - i'w_2\} < 0$*,* i*,* i' *defined in the previous statements of theorem, then* $r_w(j) = 0$*.*

The following theorem proves that for obtaining the values of $r_w(n)$ it is sufficient to apply theorem 1 only for the case $n < w_1w_2$ and then a recursive calculus can be performed.

Theorem 2 *With the notations from theorem 1, if* $n \geq w_1w_2$*, then*

$$r_w(n) = r_w(n \ mod \ w_1w_2) + n \ div \ w_1w_2, \qquad (9)$$

with $n \ div \ w_1w_2 = \left[\frac{n}{w_1w_2}\right]$ *and* $[\cdot]$ *the integer part function.*

Proof: Let be $n = w_1 \cdot w_2 \cdot q + p$, that is $q = n \ div \ w_1 \cdot w_2$ and $p = n \ mod \ w_1 \cdot w_2 < w_1 \cdot w_2$.

1. If $n = c \cdot w_1$, then $p = w_1 \cdot c_1$, that is $c = w_1(w_2 \cdot q + c_1)$. From theorem 1 we obtain :

$$r_w(n) = q + \left(1 + \left[\frac{c_1}{w_2}\right]\right) = q + r_w(p)$$

2. If $n = c \cdot w_2$, then $p = w_2 \cdot c_2$, that is $c = w_2(w_1 \cdot q + c_2)$. From theorem 1 we obtain :

$$r_w(n) = q + \left(1 + \left[\frac{c_2}{w_1}\right]\right) = q + r_w(p)$$

3. If $n \not\vdots w_1$ and $n \not\vdots w_2$, than let be $n = jw_1w_2 + i$, $i < w_1w_2$. and let be $m = (j+1)w_1w_2 + i$. We will use induction on n and the following result proved in [12]:

$$r_w(m) = r_w(n) + 1 \qquad (10)$$

□

Three particular different cases can be distinguish from these general results:

1. $w_1 = 1$, or $w_2 = 1$
2. $(w_1, w_2) = 1$
3. $(w_1, w_2) = p > 1$

The case 3 can be expressed using the case 2 and the following proposition:

Proposition 1 *([12]) Let be* $w = (w_1, w_2) \in (N^*)^2$ *a weight and* $(w_1, w_2) = p$*,* $p \in N$*, then* $\delta_w(x^\alpha) = \delta_{w'}(x^{\alpha \cdot p}),$ *with* $w' = (w'_1, w'_2)$ *and* $w'_i = \frac{w_i}{p},\ i = 1, 2.$

In practical problems, we are first interested in the case 1, be cause we will give the weight 1 for one axis (the temporal one or for the space ones). That is way we will present some results related to this case.

Theorem 3 *If* $w = (1, w_2) \in Z^2_+$*,* $w_2 > 1$*, then the dimension of the* w *- homogeneous polynomial space of degree* n *and the exponents of the monomials which generate this space, are given by*

$$r_w(n) = 1 + q \qquad (11)$$

$$A^0_{n,w} = \{(j, 0), (j - w_2, 1), \ldots, (j - qw_2, q)\}, \qquad (12)$$

with $q = \left[\frac{n}{w_2}\right]$.
If $w = (w_1, 1) \in Z^2_+$*,* $w_1 > 1$*, then*

$$r_w(n) = 1 + q$$

$$A^0_{n,w} = \{(0, j), (1, j - w_1), \ldots, (q, j - qw_1)\}. \qquad (13)$$

Proof: Let $w = (w_1, w_2)$, with $w_1 = 1$.
Then $n = n \cdot w_1$, and according theorem 1,

$$r_w(n) = \left[\frac{n}{w_2}\right] + 1 = q + 1,$$

$\alpha_2 \in \{0, \ldots, q\}$, $\alpha_1 = n - w_2\alpha_2$.
A similar proof can be made for $w_2 = 1$. □

Theorem 4 *If* $w = (1, w_2) \in Z^2_+$*,* $w_2 > 1$*, then the dimension of the* w *- polynomial space of degree* n *is*

$$d_{w,n} = dim(\Pi_{n,w}) = \frac{(q+1)(w_2q + 2r)}{2}, \qquad (14)$$

with $q = \left[\frac{n}{w_2}\right]$ *and* $r = n \ mod \ w_2$.

Proof: $dim(\Pi_{n,w}) = \sum_{j=0}^{n} dim(\Pi^0_{n,w})$.
From (12) we observe that $dim(\Pi^0_{iw_2,w}) = dim(\Pi^0_{(i+1)w_2,w}) = \ldots = dim(\Pi^0_{(i+w_2-1)w_2,w}) = i+1, \forall\, i \geq 0$. Therefore $d_w = 1\cdot w_2 + \ldots + q\cdot w_2 + (q+1)\cdot r = w_2 \cdot \frac{q(q+1)}{2} + (q+1)\cdot r$. $\square$

3 The interpolation problems

We consider, in the space of polynomial of w-degree, many interpolation problems having the conditions of type given in (8).

First, we consider the interpolation problem with the conditions:

$$\Lambda_{o,w} = \{\lambda_j(f) = f^{[j]_w}(\theta_j), \theta_j \in \Theta \subset R^2\}, \quad (15)$$

$j \in \{0,\ldots,n\}$. We want to find an interpolation polynomial with minimum w- degree. In order to do this we generalize least interpolation (see [3]) for the space of polynomials with w-degree.
We introduce the following notation:

$$< f,p >= (p(D)f)(0) = \sum_{\alpha\in N^2} \frac{D^\alpha p(0) D^\alpha f(0)}{\alpha!}. \quad (16)$$

The generating function of λ_j is:

$$\begin{aligned} \lambda^\nu_j(z) &=< \lambda_j, e_z >= e_z^{[j]_w}(\theta_j) = \\ &= \sum_{\alpha\cdot w=j} \frac{\theta_j^\alpha \cdot z^\alpha}{\alpha!} \in \Pi^0_{j,w} \end{aligned} \quad (17)$$

The spaces from (1)-(2) become

$$\begin{aligned} H_{\Lambda_{o,w}} &= span\left\{ p_j(x) = \sum_{\alpha\cdot w=j} \frac{\theta_j^\alpha \cdot x^\alpha}{\alpha!} \right\}, \quad (18) \\ H_{\Lambda_{o,w}}\downarrow_w &= span\{p_j\downarrow | p_j(x) \in H_{\Lambda_{o,w}}\} \quad (19) \end{aligned}$$

$j \in \{0,\ldots,n\}$. Obviously $H_{\Lambda_{o,w}} = H_{\Lambda_{o,w}}\downarrow_w$, because it is generated by w- homogeneous polynomials.

For any $f \in \mathcal{A}_0$, we have,

$$\lambda_j(f) = f^{[j]_w}(\theta_j) =< f, p_j >= (p_j(D)f)(0) \quad (20)$$

Let $L_{\Lambda_{o,w}}$ be the interpolation operator for the conditions (15).

Theorem 5 *The operator $L_{\Lambda_{o,w}}$ has the expression:*

$$L_{\Lambda_{o,w}} = \sum_{j=0}^{n} p_j \cdot \frac{< p_j, f >}{< p_j, p_j >}, \quad (21)$$

with p_j given in (18).

Proof: We have that $< p_j, p_i >\neq 0 \iff j = i$. The following equality holds, for all $j \in \{0,\ldots,n\}$ and $f \in \mathcal{A}_0$:

$$< L_{\Lambda_{o,w}}, p_i >=< f, p_i > \quad (22)$$

By a simple computation, we obtain:
$\lambda_j(L_{\Lambda_{o,w}}(f)) =< L_{\Lambda_{o,w}}, p_j >=< f, p_i >= f^{[j]}(\theta_j)$. $\square$

Theorem 6 *The fundamental interpolation polynomials, φ_i, $i \in \{0,\ldots,n\}$, for the interpolation scheme $(\Lambda_{o,w}, H_{\Lambda_{o,w}})$ are given by*

$$\varphi_i = \frac{p_i}{< p_i, p_i >} \quad (23)$$

Proof: $\lambda_j(\varphi_i) = \left\langle p_j, \frac{p_i}{< p_i, p_i >} \right\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol. $\square$

Next, we will considerate the case $w_1 = 1$ or $w_2 = 1$.

Proposition 2 *If in the interpolation problem with conditions (15) $w_1 = 1$ or $w_2 = 1$, then the maximum degree of the interpolation polynomial $(L_{\Lambda_{o,w}})(f)$ is n.*

Proof: The degree of the interpolation polynomial is given by the maximum degree of p_n. Taking into account the theorem 3, p_n is a linear combination of the monomials which exponents are given in (12) or (13). The maximum degree of these monomials is n. $\square$

Let observe that $d_{n,w} \neq n$. So the interpolation space falls to be $\Pi^0_{n,w}$.

We want to find a set of conditions for which $\Pi_{n,w}$ is an interpolation space. A necessary condition for this is that $d_{n,w} = \#(\Lambda)$. We consider the set of condition:

$$\Lambda_{w,\Theta} = \{\lambda_{j,k}(f) = f^{[j]_w}(\theta_{j,k})\}, \quad (24)$$

with $j \in \{0,\ldots,n = q\cdot w_2 + r\}$, $k \in \{1,\ldots d^0_{j,w}\}$, $\Theta = \{\theta_{j,k}\} \subset R^2$ a set of points having the following properties:

1. $\theta_{j,i} \neq \theta_{j,k},\ \forall i \neq k$.
2. $$\Delta_j = \left|\theta^\alpha_{j,k}\right| \neq 0, \quad (25)$$
   $j \in \{0,\ldots,n\}$, $k \in \{1,\ldots d^0_{j,w}\}$, $\alpha \in A^0_{j,w} = \{(j - i\cdot w_2, i) | i = 0,\ldots,q\}$,

Theorem 7 *The interpolation problem with conditions Λ_w, given in (24), has an unique solution in the space of polynomials of w-degree n, with $w = (1, w_2)$, $w_2 > 1$.*

Proof: Let be $p = \sum_{\alpha \in A_{n,w}} c_\alpha x^\alpha$ the interpolation polynomial. We look for its coefficients, c_α. The interpolation conditions leads to $n+1$ Cramer systems, having $d^0_{j,w}$ equations and the determinant $\Delta_j \neq 0$, $j \in \{0, \ldots, n\}$. Hence, all of these systems have an unique solution. $\square$

4 Conclusion

The new results on the polynomial spaces of w-degree, obtained in section 2 allow us to define many interpolation problems in these spaces. These interpolation problems arise from the practical problem in which both spatial and temporal conditions are imposed.

We find a set of conditions for which the interpolation problem has unique solution in the space of polynomials of w-degree n. This is important in order to use finite element methods on these spaces.

The results obtained can be generalized and developed in many directions:

1. For sets of points in R^d, with $d > 2$. Even though the theoretical results we obtained can be easily extended for $d > 2$, it is interesting to get numerical and computational details for $d > 2$ (especially for $d = 3$ and $d = 4$, cases which appear in practical problems).

2. By using other generalized degree for the polynomials.

3. By implemented the theoretical results.

These will be our further directions of study.

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