On a sequence of approximation operators constructed by means of generating functions for binomial polynomials

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Abstract: In this paper, using the umbral calculus, we introduce an sequence of linear and positive operators. Some approximation properties on given.

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1 Introduction

This section contains some basic elements of the umbral calculus (Gian–Carlo Rota and Steven Roman).

We shall be concerned with the algebra (over a field of characteristic zero) of all polynomials $p(x)$ in one variable, to be denoted by Π.

By a polynomial sequence we shall denote a sequence of polynomials $p_n(x)$, $n = 0, 1, 2, \ldots$, where $p_n(x)$ is exactly of degree n for all n.

A polynomial sequence is said to be of binomial type if it satisfies the infinite sequences of identities

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y),\ n = 0, 1, \ldots.$$

The simplest sequence of binomial type is of course x^n, but we give some nontrivial examples

1. $p_n(x) = x(x-na)^{n-1}$, (Abel)
2. $(x)^{[n,-1]} = x(x+1)\cdot\ldots\cdot(x+n-1)$, (upper-factorials)
3. $(x)^{[n,1]} = x(x-1)\cdot\ldots\cdot(x-n+1)$, (lower-factorials)

The most important shift–invariant operators are the shift operators, written E^a, that is

$$E^a p(x) = p(x+a).$$

An operator $T : \Pi \to \Pi$ which commutes with all shift operators is called a shift–invariant operator. In symbols, $TE^a = E^a T$, for all real a in the field.

We define a delta operator (E.B. Hildebrand, Gian–Carlo Rota), usually denoted by Q, as a shift–invariant operator for which Qx is a nonzero constant.

For examples:

1. $Q = D\,E^a$ (Abel operator)
2. $Q = \frac{1}{a}(I - E^{-a})$, $a \neq 0$, (backward difference operator)
3. $Q = \frac{1}{a}(E^{-a} - I)$ $a \neq 0$, (forward difference operator.)

If Q is a delta operator, then $Q\,a = 0$ for every constant a.

A polynomial sequence $p_n(x)$ is called the sequence of basic polynomials for Q if

$i)$ $p_0(x) = 1$
$ii)$ $p_n(0) = 0$ whenever $n \geq 1$
$iii)$ $Qp_n(x) = np_{n-1}(x)$, $n \geq 1$.

Every delta operator has an unique sequence of basic polynomials.

The typical example of a basic polynomial sequence is x^n, basic for the derivative operator D, $D\,p(x) = p'(x)$.

The following theorem generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

Theorem 1 *Let T be a shift–invariant operator and let Q be a delta operator with basic set $p_n(x)$.*

Then

$$T = \sum_{k\geq 0} \frac{(T\,p_k)(0)}{k!} Q^k. \tag{1}$$

In the following, we write $Q = f(D)$ where $f(t)$ is a formal power series.

An important result with generating functions for binomial polynomials is in the following theorem

Theorem 2 *Let Q be a delta operator with basic polynomials $p_n(x)$ and let $Q = f(D)$. Then there exists the inverse formal power series $f^{-1}(u)$, and*

$$\sum_{n\geq 0} \frac{p_n(x)}{n!} u^n = e^{xf^{-1}(u)}. \tag{2}$$

For example, we consider the delta operator

$$Q = \frac{1}{a}\left(E^{\varphi(a)} - I\right),$$

where I is the identity operator and $\varphi : J \to (0,1)$, $J \subset \mathbb{R}$, $0 \notin J$, is a real function with

$$\lim_{x\to 0} \varphi(x) = 0, \quad \lim_{x\to 0} \frac{\varphi(x)}{x} = 1.$$

The basic set for Q is

$$p_n(x) = \frac{a^n}{\varphi^n(a)} x(x-\varphi(a))\cdot\ldots\cdot(x-(n-1)\varphi(a)),$$

where $n \geq 1$, $p_0(x) = 1$.

We denote

$$p_n(x) = \frac{a^n}{\varphi^n(a)} \cdot (x)^{[n,\varphi(a)]},\ n\geq 1,\ p_0(x) = 1.$$

We have $f(t) = \frac{1}{a}\left(e^{\varphi(a)\cdot t} - 1\right)$ and hence

$$f^{-1}(u) = \frac{1}{\varphi(a)} \ln(1+au).$$

Using (2), we obtain

$$(1+au)^{\frac{x}{\varphi(a)}} = \sum_{k=0}^{\infty} \frac{a^k}{\varphi^k(a)} \cdot \frac{(x)^{[k,\varphi(a)]}}{k!} \cdot u^k$$

and consider the sequence of linear operators

$$(P_n f)(x) = \tag{3}$$

$$= (1+au)^{-\frac{nx}{\varphi(a)}} \sum_{k=0}^{\infty} \frac{a^k}{\varphi^k(a)} \cdot \frac{(nx)^{[k,\varphi(a)]}}{k!} u^k f\left(\frac{k}{n}\right)$$

where $x \geq 0$, $f : [0,\infty) \to \mathbb{R}$.

If we impose that

$$(P_n e_1)(x) = x,\ e_k(x) = x^k,\ k\in\mathbb{N}$$

we find

$$(P_n e_1)(x) =$$

$$= (1+au)^{-\frac{nx}{\varphi(a)}} \sum_{k=0}^{\infty} \frac{a^k}{\varphi^k(a)} \cdot \frac{(nx)^{[k,\varphi(a)]}}{k!} u^k \cdot \frac{k}{n} =$$

$$= \frac{1}{n}(1+au)^{-\frac{nx}{\varphi(a)}} \sum_{k=1}^{\infty} \frac{a^k}{\varphi^k(a)} \cdot \frac{(nx)^{[k,\varphi(a)]}}{(k-1)!} u^k.$$

Using the identity

$$(nx)^{[k,\varphi(a)]} =$$

$$= nx(nx)^{[k-1,\varphi(a)]} - (k-1)\varphi(a)(nx)^{[k-1,\varphi(a)]}$$

we have

$$(P_n e_1)(x) =$$

$$= \frac{a}{\varphi(a)} xu(1+au)^{-\frac{nx}{\varphi(a)}} \sum_{k=1}^{\infty} \frac{a^{k-1}}{\varphi^{k-1}(a)} \cdot \frac{(nx)^{[k-1,\varphi(a)]}}{(k-1)!} u^{k-1} -$$

$$-\frac{au}{n}(1+au)^{-\frac{nx}{\varphi(a)}} \sum_{k=2}^{\infty} \frac{a^{k-1}}{\varphi^{k-1}(a)} \cdot \frac{(nx)^{[k-1,\varphi(a)]}}{(k-2)!} u^{k-1} =$$

$$= \frac{a}{\varphi(a)} xu - au(P_n e_1)(x)$$

and hence

$$u = \frac{\varphi(a)}{a(1-\varphi(a)}.$$

In the present paper we investigate the sequence of linear and positive operators defined by

$$(P_n f)(x) = \tag{4}$$

$$= (1-\varphi(a))^{\frac{nx}{\varphi(a)}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,\varphi(a)]}}{k![1-\varphi(a)]^k} f\left(\frac{k}{n}\right)$$

where $\varphi : J \to (0,1)$, $0 \notin J \subset \mathbb{R}$, $x \geq 0$, and $f \in \mathcal{C}([0,\infty))$ is a real and bounded function.

Remark 1 *For $\varphi(x) = \sin x$, $\varphi : (0,\pi) \to (0,1)$ we have*

$$\lim_{x\to 0} \varphi(x) = 0, \quad \lim_{x\to 0} \frac{\varphi(x)}{x} = 1,$$

and

$$(P_n f)(x) =$$

$$= (1-\sin a)^{\frac{nx}{\sin a}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,\sin a]}}{k!(1-\sin a)^k} f\left(\frac{k}{n}\right).$$

Remark 2 *For $\varphi(a) = a$, $\varphi : (0,1) \to (0,1)$, the delta operator is* $Q = \frac{1}{a}(E^a - I)$ *(forward difference operator) with the basic set* $p_n(x) = (x)^{[n,a]}$ *and*

$$(P_n f)(x) = (1-a)^{\frac{nx}{a}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,a]}}{k!(1-a)^k} f\left(\frac{k}{n}\right).$$

Remark 3 *Finally we wish to notice that the Mirakyan–Favard–Szász operator*

$$(M_n f)(x) = e^{-\frac{nx}{a}} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

may be obtained as a limiting case of our operators (4).

For $a \to 0$ we have

$$Q \longrightarrow D,\ (nx)^{[k,\varphi(a)]} \to (nx)^k$$

and

$$(1-\varphi(a))^{\frac{1}{\varphi(a)}} \to e^{-1}.$$

Hence $P_n \longrightarrow M_n$.

2 Approximation properties

New we study the convergence of the sequence (4).

Lemma 1 *The following identities*

$$(P_n e_0)(x) = 1,\quad (P_n e_1)(x) = x, \qquad (5)$$

$$(P_n e_2)(x) = x^2 + \frac{1-\varphi(a)}{n} x$$

are valid.

Proof: Evidently that $(P_n e_0)(x) = e_0(x)$ and $(P_n e_1)(x) = e_1(x)$.

Next

$$(P_n e_2)(x) = (1-\varphi(a))^{\frac{nx}{\varphi(a)}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,\varphi(a)]}}{k!(1-\varphi(a))^k} \frac{k^2}{n^2} =$$

$$= (1-\varphi(a))^{\frac{nx}{\varphi(a)}} \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{(nx)^{[k,\varphi(a)]}}{(k-1)!(1-\varphi(a))^k} \cdot k =$$

$$= \frac{x}{1-\varphi(a)}(P_n e_1)(x) - \frac{\varphi(a)}{1-\varphi(a)}(P_n e_2)(x) +$$

$$+ \frac{x}{n(1-\varphi(a))} - \frac{\varphi(a)}{n(1-\varphi(a))}(P_n e_1)(x).$$

Hence

$$(P_n e_2) = x^2 + \frac{1-\varphi(a)}{n} x.$$

Theorem 3 *If P_n is defined by (4) then one has*

$$\lim_{n\to\infty} P_n f = f$$

uniformly on any compact $K \subset [0,\infty)$.

Proof: By making use the identities (5) we can write

$$\lim_{n\to\infty} (P_n e_k)(x) = e_k(x),\quad k = 0,1,2$$

uniformly on any compact $K \subset [0,\infty)$.

Consequently, our assertion appears directly from the well known theorem of Bohman–Korovkin.

Theorem 4 *If P_n is defined by (4) then for each $x \geq 0$ the following inequality*

$$|\ (P_n f)(x) - f(x)\ | \leq$$

$$\leq \left(1 + \min\left(\sqrt{1-\varphi(a)}, 1-\varphi(a) + \frac{1}{3nx}\right)\right)\omega\left(f; \sqrt{\frac{x}{n}}\right)$$

holds, where $\omega(f;\delta) = \sup_{0<h\leq\delta} \sup_{x\geq 0} |\ f(x+h) - f(x)\ |$ *is the first modulus of continuity.*

Proof: We have

$$|\ (P_n f)(x) - f(x)\ | \leq$$

$$\leq (1-\varphi(a))^{\frac{nx}{\varphi(a)}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,\varphi(a)]}}{k!(1-\varphi(a))^k} \ |\ f\left(\frac{k}{n}\right) - f(x)\ |\ .$$

We consider

$$|\ f\left(\frac{k}{n}\right) - f(x)\ | \leq \sup_{x,t\geq 0} |\ f(x) - f(t)\ | =$$

$$= \omega(f; |\ x-t\ |)) \leq \left(1 + \delta^{-2}(x-t)^2\right)\omega(f;\delta). \quad (6)$$

For $|\ x-t\ | < \delta$ we have

$$\omega(f; |\ x-t\ |) < \omega(f;\delta) \leq (1 + \delta^{-2}(x-t)^2)\omega(f;\delta)$$

and the last inequality (6) is valid.

For $|\ x-t\ | \geq \delta$ we have

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta) \leq (1+\lambda^2)\omega(f;\delta),$$

where $\lambda = \delta^{-1} |\ x-t\ |$, $\lambda \geq 1$.

Next we introduce the following integral method

$$| (P_n f)(x) - f(x) | \leq$$

$$\leq (1-\varphi(a))^{\frac{nx}{\varphi(a)}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,\varphi(a)]}}{k!(1-\varphi(a))^k} \mid f\left(\frac{k}{n}\right) - f(x) \mid =$$

$$= \sum_{k=0}^{\infty} p_{n,k}^a(x) \mid f\left(\frac{k}{n}\right) - f(x) \mid =$$

$$= n \sum_{k=0}^{\infty} p_{n,k}^a(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mid f\left(\frac{k}{n}\right) - f(x) \mid dt$$

where

$$p_{n,k}^a(x) = \left(1 - \varphi(a)\right)^{\frac{nx}{\varphi(a)}} \frac{(nx)^{[k,\varphi(a)]}}{k!(1-\varphi(a))^k}. \quad (7)$$

Using Lemma 1 and the inequality (6) we have

$$| (P_n f)(x) - f(x) | \leq$$

$$\leq n \sum_{k=0}^{\infty} p_{n,k}^a(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(1 + \delta^{-2}(x-t)^2\right) \omega(f;\delta) dt =$$

$$= n\omega(f;\delta) \left(\frac{1}{n} + \delta^{-2} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \sum_{k=0}^{\infty} p_{n,k}^a(x)(x-t)^2 dt \right) =$$

$$= \omega(f;\delta) \left(1 + \delta^{-2} \left(\frac{1-\varphi}{n} x + \frac{1}{3n^2} \right) \right).$$

For $\delta = \sqrt{\frac{x}{n}}$ we obtain

$$| (P_n f)(x) - f(x) | \leq \quad (8)$$

$$\leq \left(2 - \varphi(a) + \frac{1}{3nx}\right) \omega\left(f; \sqrt{\frac{x}{n}}\right).$$

Now, we starting with the inequality

$$\mid f\left(\frac{k}{n}\right) - f(x) \mid \leq \omega\left(f; \mid \frac{k}{n} - x \mid\right)$$

using the following property of the first modulus of continuity

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta)$$

we obtain for $\lambda = \delta^{-1} \mid \frac{k}{n} - x \mid$

$$\mid f\left(\frac{k}{n}\right) - f(x) \mid \leq \omega\left(f; \mid \frac{k}{n} - x \mid\right) \leq$$

$$\leq \left(1 + \delta^{-1} \mid x - \frac{k}{n} \mid\right) \omega(f;\delta).$$

Hence

$$| (P_n f)(x) - f(x) | \leq$$

$$\leq \left(1 + \delta^{-1} \sum_{k=0}^{\infty} p_{n,k}^a(x) \mid x - \frac{k}{n} \mid\right) \omega(f;\delta)$$

where $p_{n,k}^a(x)$ is defined by (8).

From the Cauchy – Schwarz – Buniakowski inequality we have

$$\sum_{k=0}^{\infty} p_{n,k}^a(x) \mid x - \frac{k}{n} \mid \leq \left(\sum_{k=0}^{\infty} p_{n,k}^a(x) \left(x - \frac{k}{n}\right)^2 \right)^{\frac{1}{2}}.$$

But

$$\sum_{k=0}^{\infty} p_{n,k}^a(x) \left(x - \frac{k}{n}\right)^2 =$$

$$= x^2 - 2x(P_n e_1)(x) + (P_n e_2)(x) = \frac{1-\varphi(a)}{n} x$$

and hence

$$| (P_n f)(x) - f(x) | \leq \left(1 + \delta^{-1} \sqrt{\frac{1-\varphi(x)}{n} x}\right) \omega(f;\delta).$$

For $\delta = \sqrt{\frac{x}{n}}$ we obtain the following inquality

$$| (P_n f)(x) - f(x) | \leq \quad (9)$$

$$\leq \left(1 + \sqrt{1-\varphi(x)}\right) \omega\left(f; \frac{x}{n}\right).$$

From (8) and (9) it results

$$| (P_n f)(x) - f(x) | \leq \quad (10)$$

$$\leq \left(1 + \min\left(\sqrt{1-\varphi(a)}, 1 - \varphi(a) + \frac{1}{3nx}\right)\right) \omega\left(f; \sqrt{\frac{x}{n}}\right).$$

Remark 4 *For $a \to 0$ we get the well-known i-nequality for Mirakyan–Favard–Szász operator*

$$| (M_n f)(x) - f(x) | \leq 2\omega\left(f; \sqrt{\frac{x}{n}}\right). \quad (11)$$

3 Conclusion

The sequence (P_n) of linear and positive operators defined by (4) are obtained using the umbral calculus (further, using the formula for generating function of binomial polynomials) verify

$$\lim_{n\to\infty} P_n f = f$$

uniformly on any compact $K \subset [0,\infty)$ and $f \in \mathcal{C}([0,\infty))$ is a real and bounded function.

Is important the fact that the well–known sequences of approximation operators Mirakyan–Favard–Szász my be obtained as a limiting case of our operators.

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