The Assignment of Dominant Pole Domains of Control Systems with Interval Parameters

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Abstrac:-. New technique of robust pole assignment based on *angle criterion* of root locus for plant with interval parameters is proposed. The proposed approach allows to assign the dominant pole domains of interval system with guaranteed stability margin and damping ratio. The other pole localization domains assigns to the required regions on the base of D – partition technique. The uncertainties of the plant are represented as interval uncertainties. In order to show the effectiveness of the proposed technique, the example of its application is given.

Keywords: - pole domain assignment, interval system, dominant pole, root locus method.

1 Introduction

The problem of controller designing for system with interval parameters was studied in many papers [1-6]. Some of the designing techniques based on postmodern theory are iterative and require powerful tools for application. Moreover, some methods (H_2, H_∞, etc.) not always give good solutions, because they require the approximation of full order controller. There are some papers with simpler approaches, but they, are based on different kinds of optimization, also iterative and requiring some extra calculations [3, 4]. Consequently, a universal design approach for fixed order control system has not been established yet.

In practice, the settling time and damping ratio are often the most important factors. It is a known fact that the performance of system is determined by two-three dominant poles, because the impact of other poles is insignificant. In the paper some techniques of fixed ordered feedback controller designing for plant with interval parameters via pole assignment are developed.

The problem of pole assignment for interval systems is considered in, but the designing technique described there also had an iterative nature [5,6].

The problem of dominant poles assignment in required points of complex plane for stationary systems has been studied in [7]. The special feature of approach offered in this paper, is an opportunity to locate non-dominant poles in the required domain and also to locate dominant poles in required points.

The proposed method based on the robust extension of approach [7]. As the coefficients of the characteristic polynomial have bounds, thus poles of the system are located in some circled domains. But for system with interval and affine uncertainty, system performance can be defined via vertex polynomials [8]. The required location of dominant and other (so-called free) poles supposes that the required bounds cannot be crossed by poles localization domains in spite of the changing of interval parameters.

2 The statement of the problem

Let the plant has a characteristic polynomial:

$$R(p) = \sum_{i=0}^{n} a_i(\bar{k}) p^i, \tag{1}$$

$\bar{k}$ - vector of regulator adjusted parameters, $a_i(\bar{k}) \in [a_{i\min}(\bar{k}), a_{i\max}(\bar{k})]$ - interval polynomial coefficients. The coefficients $a_i(\bar{k})$ are linear functions of $\bar{k}$ and formative polyhedron P_a. The vertices of P_a correspond to vertex polynomials $R^v(p)$.

The problem is to define such $k_i, i = 1, 2, \dots, r$, so the maximum value of real part of all the poles , which is related to the settling time would be less then required, the damping of system be less than required and free poles be located into required area (Fig.1.).

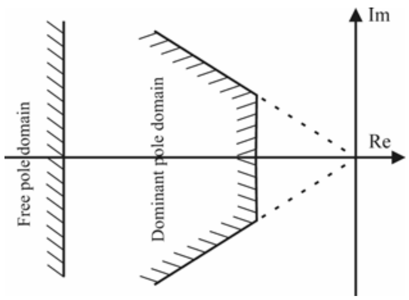

Fig.1. – Pole location domains.

The equation (1) contains interval parameters then it corresponds to family of polynomials with constant coefficients. But the worst performance of this family of stationary system can be evaluated by only one (two) polynomial. In our case we try to find one polynomial that allows evaluate maximum damping ratio and minimum stability margin. In this sense the problem is divided into two parts:

1. To define only one vertex polynomial $R^v(p)$ which roots allow to evaluate maximum damping and minimum stability margin.
2. To define the parameters of regulator to locate roots of determined vertex polynomial in required way (according to the dominant principle).

3 Vertex polynomial forming

Let polynomial (1) has *m* poles p_g, $g=\overline{1,m}$, located on the left of the pole p_0. The Θ_g denotes angle between real axis and vector, directed from p_g to p_0. There is the location of poles for $m=2$ on the fig. 2.

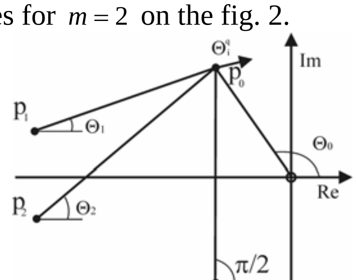

Fig. 2. – Poles and zeros location

To define vertex polynomial $R_b^v(p)$, which roots determine performance of interval system, the *angle criterion* of root locus is used. When interval coefficient a_i increases the departure angle from complex root can be found as:

$$\Theta_i^q = \pi - \left(\sum_{g=1}^{m} \Theta_g + \frac{\pi}{2} \right) + i\Theta_0 , \tag{2}$$

when a_i decreases as:

$$\Theta_i^q = -\left(\sum_{g=1}^{m} \Theta_g + \frac{\pi}{2} \right) + i\Theta_0 , \tag{3}$$

where Θ_0 – angles between real axis and vector from *i*-th zeros to p_0. Value $\frac{\pi}{2}$ was added in order to take into account root, complex conjugated to p_0. The interval value $\sum_{g=1}^{m} \Theta_g$ calculated under the *i*-th coefficient is denoted by C_i. The union of set $\sum_{g=1}^{m} \Theta_g$ for one polynomial denotes C.

$$C = C_0 \cap C_1 \cap ... \cap C_n .$$

We can formulate the condition for providing required performance by using (2) and (3). Polynomial $R_b^v(p)$ determines maximum damping ratio and minimum stability margin of interval polynomial $R(p)$ if departure angle vectors under all a_i, $i=\overline{0,n}$, are directed into the sector $\Gamma \in \left[\Theta_0, \frac{3\pi}{2} \right]$ (fig. 3). Condition described by:

$$\Theta_0 < \Theta_i^q < \frac{3\pi}{2} , \tag{4}$$

where $\Theta_i^q = i\Theta_0 - \left(C_i + \frac{\pi}{2} \right) + \Omega$ and $\Omega = 0$ or $\Omega = \pi$ is in dependence on increasing or decreasing interval parameter a_i.

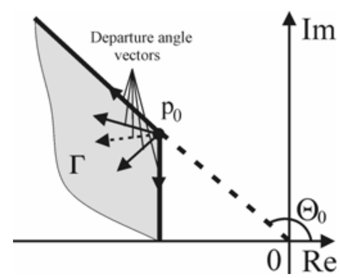

Fig. 3. – Sector Γ and departure angle vectors.

In the eq. (4) we can see that if Θ_0 is fixed, then the departure angle depends on C_i. If C_i increases then the angle Θ_i^q decreases and vice versa. Also from (4) the next condition is defined: The equality $\Theta_i^q = \Theta_0$ is held only if $C_i = \max C_i$. Using (2), (3) we define

$$\Theta_0 = i\Theta_0 - \left(\max C_i + \frac{\pi}{2} \right) + \Omega .$$

Whence the dependence between max C_i, min C_i and angle Θ_0 (that defines required sector Γ) is defined.

$$\max C_i = \Theta_0(i-1) - \frac{\pi}{2} + \Omega\,, \tag{5}$$

$$\min C_i = i\Theta_0 + \Omega\,. \tag{6}$$

It is necessary to consider dependence between C_i and free poles location in order to define the dependence between sector Γ and free poles location.

Let $p_{1,2} = -\alpha_1 \pm j\beta_1$, $p_0 = -\alpha_2 + j\beta_2$. Then

$$\sum_{g=1}^{2}\Theta_g = \operatorname{arcctg}\left(\frac{(\alpha_1-\alpha_2)^2-({\beta_2}^2-{\beta_1}^2)}{2\beta_2(\alpha_1-\alpha_2)}\right). \tag{7}$$

On the base of (7) the example of free poles domain S_c (for C less than some fixed value) is shown (fig. 4).

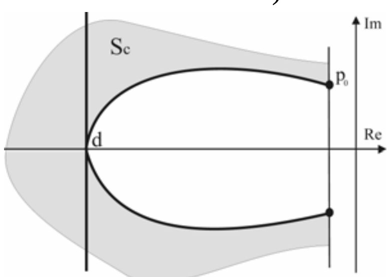

Fig. 4. – Free poles domain S_c

For every value C on the complex plane is the area of free pole location which corresponds to this C. Other words it is possible to manage value C by location of free poles into corresponded area. In eq. (5) it is defined that if C is less than some value max C then departure angle vectors are directed into required sector Γ. In the table 1 dependence of values C and limits of coefficients for the 4-th order polynomial with $\Theta_0 = \frac{2}{3}\pi$ is shown.

Table 1. Dependence between C and coefficient limits.

$\Theta_0 = \frac{2}{3}\pi$	
$C \in [0;\frac{\pi}{6}]$	$\overline{a_0}\,\underline{a_1}\,\overline{a_2}\,\overline{a_3}\,\underline{a_4}$
$C \in [\frac{\pi}{3};\frac{\pi}{2}]$	$\overline{a_0}\,\underline{a_1}\,\underline{a_2}\,\overline{a_3}\,\underline{a_4}$
$C \in [\frac{2\pi}{3};\frac{5\pi}{6}]$	$\overline{a_0}\,\overline{a_1}\,\underline{a_2}\,\overline{a_3}\,\overline{a_4}$

As the aim is to design robust controller using *dominant principle*, then consider the *leftmost* free poles location that corresponds to the smallest interval value of C. This domain is denoted by S_0. The bounds of the areas with different C_i have a complicated form, thus, the domain S_0 can be defined in simpler way, for example, by line d, parallel imaginary axis, at the left of that always lies S_0:

$$d = \frac{\beta_2}{\operatorname{tg}\left(\frac{\max C}{m}\right)} + \alpha_2\,. \tag{8}$$

From (5) and (6) we can see that $\max C_i$ cannot differ from $\min C_i$ more than $\frac{3\pi}{2} - \Theta_0$, consequently $\max C_i \in [0;\frac{3\pi}{2} - \Theta_0]$. Finally, on the base of (5), (6) the **forming polynomial** $R_b^{\nu}(p)$ **conditions** are defined:

If $\Theta_0(i-1) \in \left(\frac{\pi}{2};-\Theta_0\right)$, then $\max C_i = \Theta_0(i-1) - \frac{\pi}{2}$, $\Omega = 0$ and coefficient $a_i = \max a_i = \overline{a_i}$.

If $\Theta_0(i-1) \in \left(-\frac{\pi}{2};\pi-\Theta_0\right)$, then $\max C_i = \Theta_0(i-1) + \frac{\pi}{2}$, $\Omega = \pi$ and coefficient $a_i = \min a_i = \underline{a_i}$.

Note: If $\Theta_0(i-1) \notin \left(\frac{\pi}{2};-\Theta_0\right)$ and $\Theta_0(i-1) \notin \left(-\frac{\pi}{2};\pi-\Theta_0\right)$ then the performance of interval system can be defined by two vertex polynomial, and it is impossible to find adjusted parameters for such systems with the help of this technique.

It is necessary to use this conditions to determine Ω and minimum value of set $\max C_i$ for all $i = \overline{0,n}$ and in order to define the coefficient set and domain S_0.

4 Vertex polynomial root location technique

Let the characteristic polynomial with chosen limits of coefficients is reduced to the formula:

$$\sum_{i=1}^{r} k_i\, A_i(p) + B(p) = 0\,. \tag{9}$$

Where k_i, $i = 1, 2, 3$ – adjusted parameters, $B(p)$, $A_i(p)$, $i = 1, 2, 3$ – polynomials.

Let the boundary of free pole location is defined by the line:

$$X(j\omega) = -d + j\omega\,, \tag{10}$$

where $-\infty < \omega < \infty$ and d is found from eq. 8.

We divide the adjusted parameters into two groups. The parameter k_1 is ascribed to the first one and it is called *free* parameter. It is necessary to provide free poles location in required domain, using D-partition method.

The boundary of free poles domain (10) is represented into free coefficient space and among the obtained regions the set of free parameter is defined. Parameters k_2, k_3 are ascribed to the second group and they are called *depended*, because their calculations are based on free parameter and on the condition that the dominant poles are to take the required value. Thus, the vector of adjusted parameters $\bar{g} = (k_1, k_2, k_3)$ is divided into $\bar{g}_1 = (k_1)$ and $\bar{g}_2 = (k_2, k_3)$. Let $p = \lambda_j$, $j = 1, 2$ in eq. (9)

$$k_1 \cdot A_1(\lambda_j) + \sum_{i=2}^{r} k_i \cdot A_i(\lambda_j) + B(\lambda_j) = 0 . \tag{11}$$

Represent eq. (11) in matrix form

$$\mathbf{Q}_{11}(\boldsymbol{\lambda}) \cdot k_1 + \mathbf{Q}_{12}(\boldsymbol{\lambda}) \cdot \bar{\mathbf{g}}_2 = \mathbf{R}_1(\boldsymbol{\lambda}) , \tag{12}$$

where

$$\mathbf{Q}_{11}(\boldsymbol{\lambda}) = \begin{bmatrix} A_1(\lambda_1) \\ A_1(\lambda_2) \end{bmatrix}, \mathbf{Q}_{12}(\boldsymbol{\lambda}) = \begin{bmatrix} A_2(\lambda_1) & A_3(\lambda_1) \\ A_2(\lambda_2) & A_3(\lambda_2) \end{bmatrix}, \mathbf{g}_2 = \begin{bmatrix} k_2 \\ k_3 \end{bmatrix},$$

$$\mathbf{R}_1(\boldsymbol{\lambda}) = \begin{bmatrix} -B(\lambda_1) \\ -B(\lambda_2) \end{bmatrix}.$$

In order to locate free poles on the left of the line d (10) we substitute $p = -d + j\omega$ in eq. (9)

$$\begin{aligned} & k_1 \cdot A_1(-d + j\omega) + \sum_{i=2}^{r} k_i \cdot A_i(-d + j\omega) + \\ & + B(-d + j\omega) = 0 \end{aligned} . \tag{13}$$

Represent eq. (13) in matrix form

$$\mathbf{Q}_{21}(\omega) \cdot k_1 + \mathbf{Q}_{22}(\omega)\mathbf{g}_2 = \mathbf{R}_2(\omega) , \tag{14}$$

where

$$\mathbf{Q}_{21}(\omega) = A_1(-d + j\omega) , \mathbf{Q}_{22}(\omega) = \begin{bmatrix} A_2(-d + j\omega) & A_3(-d + j\omega) \end{bmatrix} , \mathbf{R}_2(\omega) = -B(-d + j\omega) .$$

From (12) and (14) get the system of equations to define the equation of D-partition boundary:

$$\begin{cases} \mathbf{Q}_{11}(\boldsymbol{\lambda}) \cdot k_1 + \mathbf{Q}_{12}(\boldsymbol{\lambda}) \cdot \mathbf{g}_2 = \mathbf{R}_1(\boldsymbol{\lambda}) , \\ \\ \mathbf{Q}_{21}(\omega) \cdot k_1 + \mathbf{Q}_{22}(\omega)\mathbf{g}_2 = \mathbf{R}_2(\omega) . \end{cases} \tag{15}$$

By expressing g_2 from the first equation

$$\mathbf{g}_2 = \mathbf{Q}_{12}^{-1}(\boldsymbol{\lambda}) \cdot \mathbf{R}_1(\boldsymbol{\lambda}) - \mathbf{Q}_{12}^{-1}(\boldsymbol{\lambda}) \cdot \mathbf{Q}_{11}(\boldsymbol{\lambda}) \cdot k_1 \tag{16}$$

and by substituting it in the second one, the equation of D-partition boundary is defined:

$$k_1(\omega) = \frac{\mathbf{R}_2(\omega) - \mathbf{Q}_{22}(\omega) \cdot \mathbf{Q}_{12}^{-1}(\boldsymbol{\lambda}) \cdot \mathbf{R}_1(\boldsymbol{\lambda})}{\mathbf{Q}_{21}(\omega) - \mathbf{Q}_{22}(\omega) \cdot \mathbf{Q}_{12}^{-1}(\boldsymbol{\lambda}) \cdot \mathbf{Q}_{11}(\boldsymbol{\lambda})} , \tag{17}$$

We change $\omega \in (-\infty, \infty)$, build the boundary $k_1(\omega)$ on the complex plane and define the region that corresponds to the required free poles domain. The values of k_1 that guarantee required pole assignment are located on the real axis inside the defined regions.

We substitute chosen k_1 in eq. (15) to define k_2, k_3.

5 Poles location technique

On the base of the results we have got, the interval systems poles location technique was developed.

1. The required dominant poles of interval system, which define its performance, are given.
2. Using forming polynomial $R_b^v(p)$ conditions values $\max C_i$ (for each index i) and the limit of corresponded interval coefficient are defined.
3. Out of all defined values $\max C_i$ the minimum value is chosen and on the base of (8) d is defined.
4. Polynomial (1) is reduced to formula $\sum_{i=1}^{r} k_i A_i(p) + B(p) = 0$.
5. According to the dominant poles location technique the adjusted parameters for fixed system with characteristic polynomial $R_b^v(p)$ are defined.
6. The pole placement of obtained system to check pole assignment is built.

6 Example

The PID unit has a transfer function:

$$W_\mathrm{p}(p) = \frac{k_3 p^2 + k_2 p + k_1}{p} .$$

The transfer function of plant is:

$$W_0(p) = \frac{1}{p^3 + a_3 p^2 + a_2 p + a_1} .$$

The characteristic polynomial of closed loop system is:

$$p^4 + a_3 p^3 + (a_2 + k_3) p^2 + (a_1 + k_2) p + k_1 = 0 ,$$

where

$a_3 = [18;20]$,

$a_2 = [180;200]$,

$a_1 = [1024;1026]$.

Let the root with coordinates $p_0 = -1 + j1{,}732$ defines maximum damping ratio ($\Theta_0 = \frac{2}{3}\pi$) and minimum stability margin of system.

By using the **forming polynomial** $R_b^v(p)$ **conditions** the limits of interval coefficients $\overline{a_0}\,\underline{a_1}\,\overline{a_2}\,\overline{a_3}$ are defined. From equation (8) d=4 is obtained. But according to the dominant principle let d=7. According to the dominant poles location technique the adjusted parameters for $R_b^v(p)$ are defined:

$k_1 = 500$,

$k_2 = -702$,

$k_3 = -35$.

There is dominant pole domain on the fig. 5

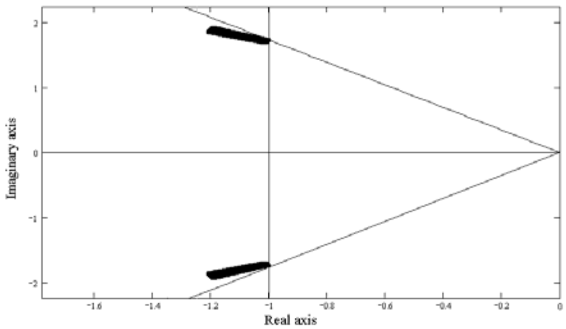

Fig. 5. – Dominant poles domain

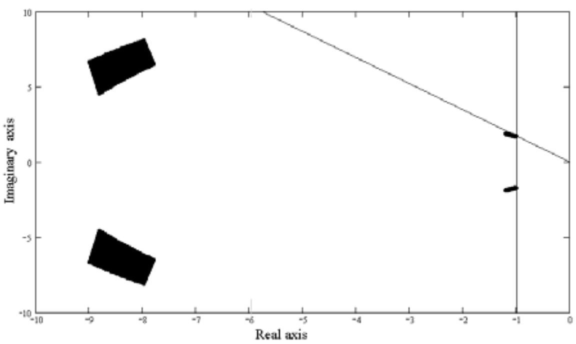

Fig. 6. – Pole placement

7 Conclusion

In this paper, we have proposed a new design technique for robust pole assignment based on angle criterion of root locus for plant with interval parameters and D – partition technique. The proposed technique was defined as two problems: the first, to find only one vertex polynomial of polynomial family, formed by interval polynomial, which defines performance of system and second, to assign the roots of this polynomial according to the obtained requirements. The advantages of the technique are: it is possible to evaluate the robustness of system under adjusted parameter perturbations by analyzing D-partition regions, the algorithm simplicity and absence of any computational complexity. The disadvantage of this technique is that the roots' behavior after them real axis touching have not been taken into account.

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