Two algorithms for decomposition of a numerical semigroup as an intersection of irreducibles

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Abstract: Every numerical semigroup S admits a decomposition $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible (that is, S_i is symmetric or pseudo-symmetric) for all i. We compare two different algorithms to obtain a numerical semigroup as intersection of irreducibles numerical semigroups.

Key–Words: numerical semigroup, symmetric and pseudo-symmetric numerical semigroup, irreducible numerical semigroup, Frobenius number.

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1 Introduction

A **numerical semigroup** is a subset S of non-negative integers $\mathbb{N}$ which contains the zero, is closed under addition and generates $\mathbb{Z}$ as a group (here $\mathbb{N}$ and $\mathbb{Z}$ denote the set nonnegative integers and the set of the integers, respectively). The greatest integer not belonging to S is called the **Frobenius number** of S, usually denoted by $\mathrm{g}(S)$. Moreover, S admits a unique minimal system of generators $\{s_1 < \cdots < s_p\}$ (that is, $S = \{\sum_{i=1}^{p} a_i s_i \mid a_1, \ldots, a_p \in \mathbb{N}\}$ and no proper subset of $\{s_1, \ldots, s_p\}$ generates S). The integers s_1 and p are known as the **multiplicity** and **embedding dimension** of S, and they are denoted by $\mathrm{m}(S)$ and $\mu(S)$, respectively.

Given $s_1 \in S \setminus \{0\}$, the **Apéry set** (called so after [1]) of S with respect to s_1 is defined by $\mathrm{Ap}(S, s_1) = \{s \in S \mid s - s_1 \notin S\}$ and it can be proved that if we choose $w(i)$ to be the least element in S congruent with i modulo s_1, then $\mathrm{Ap}(S, s_1) = \{0, w(1), \ldots, w(n-1)\}$. The set $\mathrm{Ap}(S, s_1)$ determines completely the semigroup S, since $S = \langle \mathrm{Ap}(S, s_1) \cup \{s_1\} \rangle$ (here $\langle A \rangle$ denotes the monoid generated by A). Moreover, $\mathrm{Ap}(S, s_1)$ contains in general more information that an arbitrary set of generators of S; for instance, $\mathrm{g}(S) = \max(\mathrm{Ap}(S, s_1)) - s_1$.

We say that a numerical semigroup is **irreducible** if it can not be expressed as an intersection of two numerical semigroups containing it properly. From [2] and [3] we can deduce that the class of irreducible numerical semigroups with odd (respectively even) Frobenius number is the same that the class of **symmetric** (respectively **pseudo-symmetric**) numerical semigroups. This kind of numerical semigroups have been widely studied in literature not only from the semigroupist point of view but also by their applications in Ring Theory. In [2] it is show that the semigroup ring associated to an irreducible numerical semigroup is Gorestein or Kunz if the Frobenius number is odd or even, respectively.

2 Preliminaries

Let S be a numerical semigroup. We say that an element $x \in \mathbb{Z}$ is a **pseudo-Frobenius number** of S if $x \notin S$ but $x + s \in S$ for all $s \in S \setminus 0$. We denote by $\mathrm{Pg(S)}$ the set of pseudo-Frobenius numbers of S.

We define in S the following partial order: $a \leq_S b$ if $b - a \in S$.

In [3, Proposition 7] is proved the following result showing the connection between the pseudo-Frobenius number and the Apéry set of n in S.

Lemma 1 *If S is a numerical semigroup, $s_1 \in S \setminus \{0\}$ and $\{w_{i1}, \ldots, w_{it}\} = \mathrm{maximals}_{\leq_S} \mathrm{Ap}(S, s_1)$, then* $\mathrm{Pg(S)} = \{w_{i1} - s_1, \ldots, w_{it} - s_1\}$.

Given a numerical semigroup S, denote by

$$\mathrm{H}(S) = \mathbb{N} \setminus S$$

$$\mathrm{EH}(S) = \{x \in \mathrm{H}(S) \,:\, 2x \in S, x + s \in S \text{ for all } s \in S \setminus \{0\}\}.$$

The set $\mathrm{EH}(S)$ is a subset of $\mathrm{Pg(S)}$ and thus $\#\mathrm{EH}(S) \leq \#PG(S) \leq m(S) - 1$. Using this definition, it easy to prove the next result, which describes those elements that added to a numerical semigroup yield a numerical semigroup.

Proposition 2 *Let S be a numerical semigroup and $x \notin S$. Then $x \in \mathrm{EH}(S)$ if only if $S \cup \{x\}$ is a numerical semigroup.*

Every numerical semigroup containing properly the numerical semigroup S must contain an element of $\mathrm{EH}(S)$. The idea is once you have an numerical semigroup S adding only an element we get news numerical semigroups $S \cup \{x_1\}, \ldots, S \cup \{x_r\}$ numerical semigroups, such that $\{x_1, \ldots, x_r\} = \mathrm{EH}(S)$. Thus we can compute a finite family of numerical semigroups that contain S, denote it by $\mathcal{V}(S)$.

In [5] it is presented that S is irreducible if and only if S is maximal in the set of numerical semigroups not containing $\mathrm{g}(S)$. Then we get the following result:

Corollary 3 *A numerical semigroup S is irreducible if and only if* $\#\mathrm{EH}(S) = 1$.

Using the above results from S we can compute $\mathcal{V}(S)$ and thus $S = \bigcap_i S_i$, with $S_i \in \mathcal{V}(S)$ and S_i irreducibles.

Proposition 4 *Let S be a numerical semigroup. Then*

$$S = S_1 \cap \ldots \cap S_n.$$

such that $S_1, \ldots, S_n$ are the minimal irreducible elements in $\mathcal{V}(S)$.

Our objective is to compare two different ways to obtain a semigroup S as intersection of irreducible semigroups. This algorithm is presented in [8] and it needs to construct the set $\mathrm{EH}(S)$. We start by describing two different algorithms to compute the set $EH(S)$. Suppose that $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$ is a semigroup represented as a set starting at 0 and has all elements of S until $s_r = \mathrm{g}(S) + 1$. From the definition we can easily see that the set $EH(S)$ is finite.

Algorithm EH 1

INPUT: A semigroup $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$

1. Compute the set $H(S) = \mathbb{N} \setminus S$.
2. Compute the set $D(S) = \{x \in H(S) : 2x \in S\}$
3. Compute the set $\mathrm{EH}(S)$ by checking if $x + s \in S$ for all $x \in D(S)$ and $s \in S$.

OUTPUT: The set $\mathrm{EH}(S)$.

Algorithm EH 2

INPUT: A semigroup $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$

1. Compute $\mathrm{Ap}(S, s_1) = \{Ap_1, \ldots, Ap_{s_1}\}$, Apéry set of S
2. Compute the set $E(S) = \max(\mathrm{Ap}(S))$ with respect to the partial order $<$ in $\mathrm{Ap}(S))$.
3. Compute $PGS(S)$, the pseudo Frobenius numbers of S.
4. Compute $\mathrm{EH}(S) = \{x \in PGS(S) : 2x \in S\}$

OUTPUT: The set $EH(S)$.

The main algorithm is the following:

Algorithm Intersection 1/2

INPUT: A semigroup $S = \{0, s_1, s_2, \ldots, s_r, \longrightarrow\}$

1. Set $R = \{\}$ and $E = \{\}$.
2. Compute $\mathrm{EH}(S) = \{e_1, \ldots, e_p\}$, using algorithm EH 1/2.
3. If $p = 1$ then $RF(S) = \{S\}$ and goto step 10.
4. Set $R_i = S \cup \{e_i\}$ for $i = 1, \ldots, p$ and $R = R \cup \{R_i\}$
5. Set $j = 1$ and $t = p$
6. Compute $\mathrm{EH}(R_j) = \{e_{j,1}, \ldots, e_{j,p_j}\}$ and set $E = E \cup \{EH(R_j)\}$
7. Set $R_{\#R+1} = R_j \cup \{e_{j,k}\}$ for $k = 1, \ldots, p_j$ and $R = R \cup \{R_{\#R+1}\}$
8. If $t \neq \#R$ then set $t = t + 1$, $j = j + 1$ and goto step 6.
9. Set $RF(S) = \{R_{i_1}, \ldots R_{i_q}\}$ where q is minimal such that $S = \bigcap_{X \in RF(S)} X$.
10. Return $RF(S)$.

OUTPUT: A list $RF(S)$ of semigroups such that $S = \bigcap_{X \in RF(S)} X$.

Proposition 5 *[8] Let S be a semigroup. Algorithm Intersection 1/2 computes a minimal set of semigroups which intersection is S.*

3 Complexity

The complexity of these algorithms will be expressed as function of $\mathrm{g}(S)$, $s_1, \ldots, s_p$ the set of generators of S and the size of the tree of semigroups.

The semigroup S is given by its generators so we have complexity $O(g(S))$ to write S in the forme described in section 2. To compute the Apéry set of S we have again complexity $O(g(S))$.

The complexity of Algorithm EH 1

First we compute $D(S)$ with complexity $O(g(S) - \frac{s_1}{2})$. Now to compute $\mathrm{EH}(S)$ we must test if $x - s \in S$ for all $s \in S$ and $x \in D(S)$ so we achieve this with complexity $O(g(S)^2 - \frac{s_1}{2} g(S))$. We conclude that the complexity of Algorithm EH 1 is $O(g(S) - \frac{s_1}{2} + g(S)^2 - \frac{s_1}{2} g(S)) = O(g(S)^2 - \frac{s_1}{2} g(S))$

The complexity of Algorithm EH 2

First we compute $Ap(S, s_1)$ with complexity $O(g(S) + s_1)$. We have that $\sharp Ap(S, s_1) = s_1$ and so the complexity of ordering this set, to compute $E(S)$, is $O(s_1(s_1 - 1)) = O(s_1^2)$. The set $PGS(S)$ is computed with $O(\sharp Ap(S, s_1)) = O(s_1)$ complexity. Finally the set $EH(S)$ is computed with complexity $O(s_1)$. Hence the total complexity of this algorithm is $O(g(S) + s_1 + s_1^2 + s_1) = O(g(S) + s_1^2)$

Remark 6 *Note that if* $S =< s_1, s_2 >$ *then* $\mathrm{g}(S) = s_1 s_2 - (s_1 + s_2)$ *or in the case where* $S =< s_1, \ldots, s_p >$ *is a* $MED-semigroup$ *then* $\mathrm{g}(S) = s_p - s_1$ *and thus we can use this result above. We can see that in these cases the complexity of Algorithm EH 1 is greater than the complexity of Algorithm EH 2.*

The complexity of Algorithm Intersection 1/2

We start by computing $EH(S)$ (using **Algorithm EH 1/2**) with complexity $O(EH)$ described above. Then after constructing the semigroups $R_i = S \cup \{e_i\}$ we compute $EH(R_i)$ and repeat this process until there are no new semigroups that appear. This is done with complexity $O(T)O(EH)$, where T is the total number of semigroups to intersect. Finally we eliminate those which are redundant. So the final complexity of these algorithms are $O(T(g(S)^2 - \frac{s_1}{2} g(S)))$ and $O(T(g(S) + s_1^2))$ respectively.

The value of T is not predictable. Meaning that we do not know any upper bound for it because it arises from a tree structure (see [8]). We will indicate in the experimental results the maximum value of T for each set of tested semigroups.

4 Experimental results

In order to test the efficiency of both algorithms we defined 200 random semigroups with 3 up to 10 generators bounded by 100, 200 and 300. We computed the maximum running time (MRT) of each algorithm and the overall average running time (ART). The results (given in seconds) are summarized in the following tables:

- For generators with values up to 100:

Generators ;	3	4	5	6	7	8	9	10
MRT for Alg 1	16.3580	14.5800	13.5790	5.2340	5.9220	3.1560	5.5320	5.1710
MRT for Alg 2	16.3140	14.3740	13.2650	5.1410	5.7670	3.0470	5.4200	5.0790
ART for Alg 1	1.2381	0.9071	0.7167	0.3954	0.5231	0.2343	0.2497	0.2343
ART for Alg 2	1.2228	0.8918	0.7126	0.3877	0.5135	0.2275	0.2457	0.2315
Max T	11	23	22	20	24	22	24	25
Average T	4.290	5.750	6.945	6.305	7.725	5.970	6.040	5.895

- For generators with values up to 200:

Generators	3	4	5	6	7	8	9	10
MRT for Alg 1	183.286	116.424	107.250	73.407	57.735	162.221	35.420	50.720
MRT for Alg 2	184.296	116.139	107.283	72.641	57.216	162.093	35.704	50.061
ART for Alg 1	15.808	13.261	9.892	7.004	4.923	6.405	2.772	2.460
ART for Alg 2	15.804	13.116	9.843	6.921	4.865	6.316	2.736	2.421
Max T	11	24	34	38	37	45	33	41
Average T	4.930	9.710	11.105	12.540	11.860	12.595	10.695	9.960

- For generators with values up to 300:

Generators	3	4	5	6	7	8	9	10
MRT for Alg 1	1139.221	561.259	456.063	346.933	798.463	321.449	100.201	275.079
MRT for Alg 2	1153.356	563.537	459.562	346.766	653.346	351.020	99.637	274.061
ART for Alg 1	65.575	43.807	35.525	27.618	31.366	17.984	10.404	12.120
ART for Alg 2	63.842	43.675	35.420	27.601	30.389	17.795	10.270	12.005
Max T	11	36	40	42	55	51	42	55
Average T	5.540	10.790	14.750	14.970	17.370	16.620	15.430	14.850

5 Conclusion

The experimental data show us two different things. The first one is that Algorithm 2 is in practice faster then Algorithm 1,(comparing the corresponding ART). The second is that, surprisingly, it is only slightly faster, indeed the difference between the corresponding ART is quite small (approximately around 0.5%). The worst case scenario complexity, of the two algorithms, are not comparable in general. This happens because there are no known relations between de frobenius number and the multiplicity of a semigroup. But for the particular semigroups, presented in section 3 remark 6, this relation is known and hence we are able to compare them.

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