Control of a Class of Hybrid Systems with Combined Continuous and Discrete Objectives

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Abstract: -This paper deals with the control of a class of perfectly modelled linear hybrid systems which consist of two, in general coupled, subsystems one being continuous -time while the second one is digital. The control objective has a double nature and it consists of the achievement of separate continuous-time and discrete-time model- matching objectives with respect to two predefined stable reference models. It is achieved by synthesizing a dynamic hybrid controller consisting of a continuous subcontroller and a discrete one. Each of those controllers has its own control objective.

Key-Words: – Hybrid systems, control laws, stability, model-following objectives

1. Introduction

Hybrid systems have received important attention in the last decades see, for instance, [2]-[5], [11-13]. In particular, the optimization of inputs and the fundamental properties of such systems have received attention in [2] and the multirate sampling of such systems has been studied in [10] and [3]. The importance of those systems arises from the fact that continuous and digital subsystems usually operate in a combined and integrated fashion. Another important reason to deal with such systems is that it becomes sometimes suitable the use of either discrete-time or digital controllers for continuous plants by technological implementability reasons, [8] -[9] .In this paper, a wide class of linear hybrid systems proposed in [2]and also dealt with in [3] is considered in the context of model-matchingdesigns. Such systems are characterized by the continuous substate being forced by both the current input in continuous time and its sampled value at the last preceding sampling instant as well. *The objective of this paper is the design of an hybrid controller that allows the hybrid plant to achieve , in genera l, separate continuous - time and discrete -time model- following objectives in the perfectly modelled situation.* In this way, the continuous-time and discrete-time closed-loop dynamics can be separately designed through the synthesis of two subcontrollers which give together the overall , in general, hybrid controller. The subcontroller designed for accomplishing with the discrete-time control objective has a discrete-time nature while that designed to accomplish with the continuous-time objective is of a mixed continuous-time and discrete-time nature. Several particular cases which are included in the general framework are for instance:

(a) The choice of only a continuous-time reference model. Thus, its digital transfer function is used as discrete model for controller synthesis at sampling instants.
(b) The use of only a discrete-time reference model under a piecewise constant plant input inbetween sampling instants. In such a case, the overall scheme becomes a discrete-time one.
(c) The use of the discrete-time reference model for periodic testing of the current closed-loop performance designed for a continuous-time reference dynamics. If the test fails then the continuous-time objective can be on - line modified in terms of re-adjustment of the input to the (continuous-time) reference model or high-frequency gain re-adjustment to modify either the transient reference signal or the steady-state reference set point. Each subcontroller is designed for the achievement of the corresponding model-following objective in the absence of plant unmodelled dynamics . Also, as a part of the design , each subcontroller generates a compensating signal to annihilate the coupling signals generated from the continuous signals to the discretized output , for the

discrete-time control objective , or viceversa. when dealing with the continuous control objective. Such coupling signals are inherent to the structure of the open-loop hybrid plant . Finally, the overall controller is robust against a class of unmodelled dynamics and uniformly bounded state and measurement noises.

2. Hybrid plant description

2.1 The hybrid plant

Consider the next single-input single-output hybrid linear system (Kabamba and Hara; De la Sen 1996):

$$\dot{x}_c(t)=A_c x_c(t)+A_{cs} x_c[k]+A_{cd} x_d[k] + b_c u(t)+b_{cs} u[k]$$
$$x_d[k+1]=A_d x_d[k]+A_{ds} x_c[k]+b_c u(t)+b_{cs} u[k]$$
$$y(t)=c_c^T x_c(t)+c_{cs}^T x_c[k]+c_d^T x_d[k]+d_c u(t)+d_d u[k] \qquad (1)$$

for $t \in [kT , (k+1) T)$; all integer $k \geq 0$, with T being the sampling period , where $x_c(.)$ and $x_d[.]$ are , respectively, the n_c and n_d continuous and digital subvectors and u (.) a nd y (.) are the scalar input and output. The continuous time argument is denoted by '(t) ' while the discrete time argument is denoted by ' [k] ' and the associated continuous and digital variables are denoted correspondingly . Thus, a continuous variable at sampling instants is denoted in the same way as a digital variable so that $x_c[k] =x_c (kT)$ and u [k] = u (kT) in (1). In that way , there is no distinction in the treatment of digital and time-discretized variables. The orders of all the real constant matrices in (1) agree with the dimensions of the substates and scalar input and output.

2.2. Description of (1) at sampling instants

The input / output solution of (1)at sampling instants is given by the ARMA- model :

$$Q_d(q)\,y[k]=P_d(q)\,u[k]+Q_d(q)\,(c^T\omega[k]) \qquad (2)$$

for all integer $k \geq 0$, where $c^T=c_c^T+c_{cs}^T$, $Q_d(q)$ and $P_d(q)$ are polynomials of real coefficients defined in Appendix A (see eqns. (A.4) and (A.6)) of degree $n = n_c + n_d$ and q is the one-step -ahead shift operator , i . e . , q y [k]=y[k+1] ; q u [k]=u [k+1] and q ω [k] = ω [k+1]. The ARMA - model (2) is obtained from the extended discrete-time system of state $x[k]=[x_c^T[k], x_d^T[k]]^T$ obtained from (1).

2.3. Description of (1) inbetwwen sampling instants

The input / output differential-difference relationship for (1) inbetween sampling instants is given by

$$Q_c(D)Q_d(q)\,y(t)= P_c(D)Q_d(q)\,u(t)+Q_c(D)\left\{ N_{cd}^{u}(D,q)\,u[k]+N_{cd}^{\omega^T}(D,q)\,\omega[k] \right\} \qquad (3)$$

for $t \in [kT , (k+1) T)$; all integer $k \geq 0$, with q and D being the one-step ahead time -shift and time-derivative defined by q v(t) = v (t + T) and $\dot{v}(t)=Dv(t)$, respectively , for any differentiable signal v (t) in the continuous-time argument t , where $Q_d(q)$ and $P_d(q)$ are the polynomials in (2) while $Q_c(q)$ and $P_c(q)$ are polynomials of degree n_c and N_{cd}^{u} and N_{cd}^{ω} are a scalar polynomial and a two-variable n_c - polynomial matrix which have been obtained from the above parameters but the parametrical definition and its development are omitted by space reasons. Note that the term in brackets in the right - hand- side of (3) is a coupling signal from the digital substate and discretized input to the continuous subsystem of (1). The description (3) is obtained from an extended hybrid system of continuous- time substate $x_c(t)$ and the discrete- time substate $x[k]=[x_c^T[k], x_d^T[k]]^T$ used for obtaining (2) at sampling instants . The next simple descriptive example illustrates the decomposition in continuous / discrete (or digital) state - variables of an input / output linear mapping involving the operators D and q as it occurs in the general description of (3).

Example. Consider the input/ output linear mapping $v(t) = H_1(D) H_3(q)\,\delta[k]+H_2(D)\,\upsilon(t)$ driven by the discrete input δ [k] and the continuous one υ(t) where $H_1(D)=\frac{D+a}{D+b}$; $H_2(D)=\frac{1}{D+c}$; $H_3(q)=\frac{q+1}{q+2}$ with a , b and c being real constants. Define now two continuous-time variables $v_1(t)$ and $v_2(t)$ and a digital variable $\delta_1[k]$ given by the dynamics $v_1(t) = \frac{D+a}{D+b}\,\delta_1[k]$, $v_2(t) =\frac{1}{D+c}\,\upsilon(t)$ and $v_3[k] = \frac{q+1}{q+2}\,\delta[k]$. The state-space is:

$$v(t)=v_1(t)+v_2(t)+v_3[k]$$
$$\dot{v}_1'(t)=-bv_1'(t)+(a-b)v_3[k]$$
$$v_1(t)=v_1'(t)+v_3[k]$$
$$\dot{v}_2(t)=-cv_2(t)+\upsilon(t);$$
$$v_3[k]=-2\,v_3[k-1]+\delta[k]+\delta[k-1]$$

subject to initial conditions $v_i(0) = v_{i0}$ (i = 1, 2) and $v_3[0] = v_{30}$. □

Remark 1 . The description of (3) also describes eqns. 1 at sampling instants and results to be

$$Q_c(D)Q_d(q)y[k]=[P_c(D)Q_d(D)+Q_c(D)N^{u}_{cd}(D,q)]u[k]+Q_c(D)N^{\omega^T}_{cd}(D,q)\omega[k]$$

whose discrete-time solution is (2). Note through a comparison with (3) that the parametrization of the differential- difference solution to (1) becomes modified at sampling instants with respect to the intersample parametrization since additive terms involving the sampled continuous substate and sampled input result from the plant parametrization given by (1) at sampling instants. □

2.4 Global exponential stability conditions for the open-loop plant

The global exponential stability of the unforced system (1) is only dependent on the stability of the *A* -matrix defined by

$$\begin{bmatrix} e^{A_c T}[I+(\int_0^T e^{-A_c\tau}d\tau)A_{cd}] & e^{A_c T}(\int_0^T e^{-A_c\tau}d\tau)A_d \\ A_{ds} & A_d \end{bmatrix} \quad (4)$$

obtained after omitted calculations . This follows from building the extended unforced discrete dynamics $x[k+1] = A\,x[k]$ with $x[k] = (x_c^T[k], x_d^T[k])^T$. Thus, the continuous - time solution of the continuous substate in (1) satisfies :

$$x_c(kT+\tau)=[e^{A_c\tau}(I+\int_0^{\tau} e^{-A_c\tau'}d\tau')A_{cs}) \;,\; e^{A_c\tau}(\int_0^{\tau} e^{-A_c\tau'}d\tau')A_{cd}]x[k]$$

all integer $k \geq 0$ and all real $\tau \in [0, T)$. Thus , if A is strictly Hurwitzian , then $x_d[k] \to 0$, $x_c[k] \to 0$ and $x_c(kT+\tau) \to 0$ exponentially fast as $k \to \infty$ for all $\tau \in [0, T)$ and for any bounded real constant $x[0]$. *The next result , whose proof is omitted, is concerned with the stability of the A - matrix under that of A_c and A_d provided that the coupling signals between continuous and discretized (or , indistinctly, digital) variables are sufficiently small.*

Proposition 1 . Assume that A_c and A_d are strictly Hurwitzian with their maximum eigenvalues satisfying :

$$e^{-\rho' T} \leq \left|\lambda_{max}(e^{A_c T})\right| \leq e^{-\rho T} (\text{i.e.}, -\rho \leq \lambda_{max}(A_c) \leq -\rho')$$

and $\left|\lambda_{max}(A_d)\right| \leq e^{-\rho T}$ for some positive real constants ρ and ρ' with $\rho' \geq \rho$ .Thus, the open-loop unforced plant is globally exponentially stable if

$$\left|\lambda_{max}(\begin{bmatrix} A_{cs} & A_d \\ A_{ds} & I \end{bmatrix})\right| < Min(e^{\rho T}-1, \frac{\rho'(e^{\rho T}-1)}{e^{\rho' T}-1}).$$

3. Controller synthesis

3.1 General design philosophy and Assumptions

The controller to be synthesized will consist of two subcontrollers each one being designed to satisfy a different (respectively, continuous-time or discrete - time) control objective , namely :

Objective 1 : $u[k] = u[kT]$ is generated in such a way that a prescribed stable discrete reference model of transfer function $W_{md}(q)$ is matched at sampling instants. A discrete subcontroller (Subcontroller 1) which will be then synthesized accomplishes with this control objective. As a part of the design , the coupling signal in (2) from the continuous-time subsystem to the discrete - time subsystem , caused by the signal $\omega[k] = e^{A_c T}(\int_0^T e^{-A_c t}u(kT+t)dt)b_c$,that includes the contribution of the continuous-time input over one sampling period to the output at sampling instants , is annihilated by synthesizing the appropriate compensator as addressed below.

Objective 2 : $u(t)$ $(t \neq kT)$ is generated in such a way that the closed-loop system matches a prescribed stable continuous-time reference model of transfer function $W_{mc}(D)$ inbetween sampling instants. A mixed continuous / discrete subcontroller (Subcontroller 2) is synthesized to accomplish with such a control objective. As a part of the design , the couplings between the discretetized signals $u[k]$ and $\omega[k]$ and the continuous subsystem are cancelled by synthesizing the appropriate compensator as addressed below. □

Since $u[k]$ and $u(t)$, $t\in[kT, (k+1)T]$, all integer $k \geq 0$ are , in general , synthesized to satisfy two different control objectives , discontinuities of the control input at sampling instants occur in general. Also , there are input discontinuities caused by the influence in the feedback signals of the modification of the digital substate at sampling instants while it is kept constant inbetween sampling instants. When suitable, the two reference models can be appropriately related to each other in order to state the problem with a unique control objective as discussed later. Those input discontinuities translate in output discontinuities at sampling instants in the more general case when $W_{mc}(D)$ and $W_{md}(q)$ are chosen independently. The combined objective can be intuitively figured as of the actions of Subcontrollers 1- 2 synthesized to satisfy the Control Objectives 1 - 2 . There are two control channels integrated in the actuator that generate the input ' at ' and ' inbetween' sampling instants as $u(t) = u'[k]$ $(t = kT)$; $u(t) = u'(t)$ $(t \neq kT)$ Channel 1 is used to generate (inbetween sampling instants)the input for model -matching of $W_{md}(q)$ while Channel 2 is used to match $W_{mc}(D)$. Note that once Channel 1 modifies its state, it supplies $u[k]$ at sampling instants .

Assumptions

1. $P_d(q)$ and $P_c(D)$ have all their zeros in $|q|<1$ and $\mathrm{Re}(D)<0$.

2. All common zeros of $P_d(q)$ and $Q_d(q)$ (of $P_c(D)$ and $Q_c(D)$) , if any , are strictly Hurwitzian and closed - loop zeros and poles of the discrete-time (continuous- time) dynamics , i. e . , they are zero- pole cancellations of $W_{md}(q)$ in $|q|<1$ (of $W_{mc}(D)$ in $\mathrm{Re}(D)<0$). Also, the zeros of $P_c(D)$ and $P_d(q)$ which are cancelled by the controller, if any, are closed- loop poles and thus poles of $W_{mc}(D)$ and $W_{md}(q)$, respectively .

3. $W_{mc}(D)$ and $W_{md}(q)$ are proper , strictly Hurwtzian and of relative orders non less than those of $P_c(D)/Q_c(D)$ and $P_d(q)/Q_d(q)$, respectively. □

Note that Assumption 1 means that both (open-loop) discrete and continuous-time descriptions eqns. 2 and 3 are inversely stable. Assumption 2 means that if any of the discrete or continuous plant dynamics is uncontrollable (i.e., there are zero-pole cancellations) *then the associated uncontrollable modes have to be stable and closed-loop poles of the corresponding dynamics*. Note also that if $d = d_c + d_d$ (d_c) is nonzero in (1) then $P_d(q)/Q_d(q)$ ($P_c(D)/Q_c(D)$) is nonstrictly proper and then the realizability of Subcontroller 1 (Subcontroller 2) is realizable for any realizable $W_{md}(q)$ ($W_{mc}(D)$). Thus, the relative order constraint of Assumption 3 holds automatically under the realizability of the discrete-time (continuous- time) reference model guaranteed by its properness of the first partof the assumption.

3.2 Objective 1 : Synthesis of Subcontroller 1 and Generation of u [k] = u (kT)

The discrete control is designed to achieve Objective 1 when the plant (1) is known and noisy-free:

$$u[k]=\frac{G_{1d}(q)}{L_d(q)}u[k]+\frac{G_{2d}(q)}{L_d(q)}y[k] + \frac{G_{3d}^T(q)}{L_d(q)}\omega[k]+\frac{R_{1d}(q)}{L_d(q)}r_{1d}[k] \quad (5)$$

all integer $k \geq 0$. The compensating signal $r_{1d}[\cdot]$ is forwarded to the plant input from the reference model input $r_d[k]$ and $\omega[k] = \left(\int_0^T e^{A_c(T-\tau)}u(kT+\tau)d\tau\right)b_c$ according to generation laws given below . All the transfer functions in the above control law are expressed as quotients of polynomials and realizable. The above law is explicited as follows:

$$u[k]= C_{yu}^d(q)y[k]+C_{\omega u}^{d^T}(q)\,\omega[k] + C_{r_1 u}^d(q)\,r_{1d}[k] \quad (6.a)$$

where the compensator transfer functions are

$$C_{yu}^d(q)=\frac{G_{2d}(q)}{L_d(q)-G_{1d}(q)}$$

$$C_{\omega u}^d(q)=\frac{G_{3d}(q)}{L_d(q)-G_{1d}(q)}$$

$$C_{r_1 u}^d(q)=\frac{R_{1d}(q)}{L_d(q)-G_{1d}(q)L_d(q)} \quad (6.b)$$

The problem of accomplishing with Objective 1 consists of designing the polynomials $G_{id}(q)$ (i=1,2) and $R_{1d}(q)$, the polynomial vector $G_{3d}(q)$ as well as the compensating signal $r_{1d}[\cdot]$, for a given stable $L_d(q)$ so that $W_{md}(q)$ is matched if the plant is perfectly known and free of unmodelled dynamics and noise. The next result addresses the controller design :

Theorem 1. Suppose that the control law (6) is applied , $r_d[k]$ ($k \geq 0$)is the uniformly bounded reference input sequence to $W_{md}(q)$ and that the next assumptions hold :

4. Assumptions 1 - 3 hold for $P_d(q)$, $Q_d(q)$ and the poles of $W_{md}(q)$, and that all the roots of $R_{1d}(q)$ and $L_d(q)$ are in $|q|<1$. Assume also that $d_c = -d_d$ in (1) and $\deg(R_{1d}) \leq \deg(L_d - G_{1d})$.

5. $P_d(q) = \bar{Q}_d(q)P'_d(q)$ and $Q_d(q) = \bar{Q}_d(q)Q'_d(q)$ where $\bar{Q}_d(q)$ is the strictly

Hurwitzian (from Assumption 2) ma ximum common factor of $P_d(q)$ and $Q_d(q)$. Also , $P'_d(q)=P_{1d}(q)P_{2d}(q)$ with $P_{1d}(q)$ being defined by the discrete strictly Hurwitzian plant zeros (from Assumption 2) which are not plant poles and they are transmitted to the reference model $W_{md}(q) = B_{md}(q) / A_{md}(q)$.

6. $L_d(q)$ is factorized as $L_d(q) = P_{2d}(q) L'_d(q)$ in (5) .

Thus , the discrete closed - loop transfer function equalizes that of $W_{md}(q)$ provided that Subcontroller 1 and its associated compensating signal $r_{1d}[.]$ are synthesized as follows :

$$r_{1d}[k]=\frac{B_{md}(q)}{R_{1d}(q)P_d(q)}r_d[k] + \frac{L_d(q)-G_{1d}(q)}{R_{1d}(q)} M_d c^T \omega[k] \quad (7)$$

where $G_{1d}(q)=G'_{1d}(q)P_{2d}(q)$, $d = d_c + d_d$, $c = c_c + c_{cs}$ with $M_d(q)$ being an arbitrary polynomial satisfying $\deg(M_d(q)) < \deg(L_d(q) - G_{1d}(q)) - \deg(R_{1d})$, $G_{3d}(q) = -(1+R_{1d}M_d c)$, and $G'_{1d}(q)$, $G_{2d}(q)$ being polynomials which are the unique solution to the diophantine equation :

$$Q'_d(q)\, G'_{1d}(q)+P_{1d}(q)G_{2d}(q) = Q'_d(q)L'_d(q)-A''_{md}(q) \quad (8)$$

subject to the degree constraints $\deg(G_{2d}(q)) < \deg(Q'_d(q))$ or $\deg(G'_{1d}(q)) < \deg(P_{1d}(q))$ for $A''_{md}(q)$ being a polynomial satisfying the factorizations

$$A_{md}(q)=\overline{Q}_d(q)A'_{md}(q) = \overline{Q}_d(q)P_{2d}(q)A''_{md}(q) \quad (9)$$

which exist from Assumption 2 . □

Corollaries : **1 .** Theorem 1 also holds under the same assumptions if G_{3d} is a rational function and the compensating signal in the controller satisfy :

$$G_{3d}(q)=\frac{Q'_d(q)(G'_{1d}(q)-L'_d(q))c}{P_{1d}(q)} \quad (10)$$

$$r_{1d}[k] = \frac{1}{R_{1d}(q)}\frac{B'_{md}(q)}{P_{2d}(q)}r_d[k] \quad (11)$$

and all the remaining compensators of the control law remaining identical as in Theorem 1 .

2. Theorem 1 and Corollary 1 als o apply directly to the regulation case with $r_d[k] = r_{1d}[k] = 0$ with the closed-loop dynamics resulting to be $A_{md}(q)\, y[k] = 0$, all integer $k \geq 0$. □

The proof of Corollary 1 becomes direct from the application of Assumptions 3 - 6 of Theorem 1 and the use of the cancelled factors $Q_d P_{2d}$ and $G_{1d}=G'_{1d}P_{2d}$ to yield :

$$G_{3d}=\frac{\overline{Q}_d Q'_d(G_{1d}-L_d)c}{P_d} = \frac{Q'_d(G'_{1d}-L'_{1d})c}{P_{1d}} \Rightarrow C^d_{r_1 u}=\frac{G_{3d}}{L'_d-G'_{1d}}=-\frac{Q'_d}{P'_d}=-\frac{Q_d}{P_d}$$

which is nonstrictly proper and stable since P_d is strictly Hurwitzian and $d_c \neq d_d$. The use of the above relationships leads to

$$[(L_d-G_{1d})Q_d-P_dG_{2d}]y[k] = Q_d(q)R_{1d}(q)r_{1d}[k] \quad (12)$$

from (7)-(11) and Corollaries 1-2 follow as Theorem 1. Corollary 2 follows when $r_d[k] \equiv 0$. □

Note that the main difference between the design of Theorem 1 and Corollary 1 is the choice of the compensator $C^d_{r_1 d}(q)$ in (6). In Theorem 1 , this proper compensator of high - frequency being $-d^{-1}c = -(d_c + d_d)^{-1}(c_c + c_{cs})$ which cancels the high- frequency gain of $\frac{G'_{1d}-L'_d}{P_{1d}}Q'_d c$. Thus , the closed-loop dynamics does not depend on $\omega[k]$ but on $\omega[k-1]$ and $M_d(q)$ is kept arbitrary. *However, the decomposition of all the transfer functions from the components of ω [k] to u [k] in Corollary 1 with their high- frequency gains being cancelled is not used*. The synthesis mechanism in that case is the choice of $G_{3d}(q)$ such that the transfer function from $\omega[k]$ to $u[k]$ is cancelled.

3. 3 Objective 2 : Synthesis of Subcontroller 2 and generation of u (t) (t ≠ k T)

The next control law is designed for the achievement of Objective 2 when the known plant is perfectly modelled and free- noise and has the following implicit structure :

$$u(t)=\frac{G_{1c}(D,q)}{L_c(D,q)}u(t)+\frac{G_{2c}(D,q)}{L_c(D,q)}y(t)+\frac{G_{3c}(D,q)}{L_c(D,q)}u[k]+\frac{G_{4c}^T(D,q)}{L_c(D,q)}\omega[k]+\frac{R_{1c}(D)}{L_c(D,q)}r_{1c}(t) \qquad (13)$$

for all $t \in (kT, (k+1)T)$ and all integer $k \geq 0$, with $r_{1c}(t)$ being a compensating signal to be generated as a part of the controller design and $L_c(D, q)$ being a strictlty Hurwitzian two-variable polynomial. The various filters are formed by two variable polynomials and the associated hybrid realizations can be obtained as addressed in the given example. The above control law becomes explicited as follows :

$$u(t)=C_{yu}^{c}(D,q)y(t)+C_{uu}^{c}(D,q)u[k]+C_{\omega u}^{c^T}(D,q)\omega[k]+C_{r_1u}^{c}(D,q)r_{1c}[k] \qquad (14)$$

for all $t \in (kT, (k+1)T)$ and all integer $k \geq 0$ with

$$C_{yu}^{c}(D,q)=\frac{G_{2c}(D,q)}{L_c(D,q)-G_{1c}(D,q)} \qquad (15)$$

$$C_{uu}^{c}(D,q)=\frac{G_{3c}(D,q)}{L_c(D,q)-G_{1c}(D,q)} \qquad (16)$$

$$C_{\omega u}^{c}(D,q)=\frac{G_{4c}(D,q)}{L_c(D,q)-G_{1c}(D,q)} \qquad (17)$$

$$C_{r_1u}^{c}(D,q)=\frac{R_{1c}(D)}{L_c(D,q)-G_{1c}(D,q)} \qquad (18)$$

Note that the compensators of (15)-(18) are dependent on D and q because of structure of (3) . The problem of fulfilling Objective 2 consists of synthesizing (14), subject to (15)-(18), as well as the compensating signal $r_{1c}(.)$ as addressed in the next result which applies the philosophy of Theorem 1 and Corollary 1 to the problem of model-matching of the continuous reference model . In the following, the degree of two-variable polynomials with respect to one of the variables is denoted with the corresponding subscript.

Theorem 2. Suppose that $r_c(t)$ is the uniformly bounded reference input to W_{mc} D) and that the next assumptions hold :

7. Assumptions 1 - 3 hold for $P_d(q)$, $Q_d(q)$, $P_c(D)$ and $Q_c(D)$ and that the poles of $W_{mc}(D)$ and all the roots of $R_{1c}(D)$ and $L_c(D)$ are in $\mathrm{Re}(D)<0$. Assume also that $d_c \neq -d_d$.

8. $P_c(D)$ admits the polynomial factorization $\overline{Q}_c(D)\,P_{1c}(D)\,P_{2c}(D)$ where $\overline{Q}_c(D)$ includes the (stable) common roots of $P_c(D)$ and $Q_c(D)$, $P_{1c}(D)$ contains eventual zeros of $P_c(D)$ transmitted from the plant to the reference model and $P_{2c}(D)$ includes the (stable) plant zeros which are closed-loop poles and controller poles .

Thus, the closed -loop dynamics is globally exponentially stable and defined by

$$A_{mc}(D)y(t)=B_{mc}(D)r_c(t) \qquad (19)$$

if the compensators in (15)-(18) and compensating signal $r_{1c}(t)$ satisfy $G_{1c}(D,q)=P_2(D,q)\,G'_{1c}(D,q)$ where $(G'_{1c}(D,q),\ G_{2c}(D,q))$ is a polynomial pair being a unique solution to the two-variable diophantine equation :

$$Q'_c(D)G'_{1c}(D,q)+P'_c(D)G_{2c}(D,q)=L'_c(D,q)-A''_{mc}(D,q) \qquad (20)$$

with $L_c(D,q)=P_{2c}(D,q)\,L'_c(D,q)$ and $A_{mc}(D,q)=\overline{Q}_c(D,q)P_{2c}(D,q)A''_{mc}(D,q)$ subject to any of the two the next degree constraints

$$\deg_D(L_c(D,q)-A_{mc}(D))\leq \deg_D(G_{1c}(D,q))$$
$$\deg_D(G_{2c}(D,q))<\deg(Q_c(D))=\deg(P_c(D)) \qquad (21.a)$$

$$\deg_D(L_c(D,q)-A_{mc}(D))\leq \deg_D(G_{2c}(D,q))$$
$$\deg_D(G_{1c}(D,q))<\deg(Q_c(D))=\deg(P_c(D)) \qquad (21.b)$$

$$G_{3c}(D,q)=\frac{G'_{1c}(D,q)-L'_c(D,q)}{\overline{Q}_c(D)P_{1c}(D)Q_d(q)}N_{cd}^{u}(D,q) \qquad (22.a)$$

$$G_{4c}(D,q)=\frac{G'_{1c}(D,q)-L'_c(D,q)}{\overline{Q}_c(D)P_{1c}(D)Q_d(q)}N_{cd}^{\omega}(D,q) \qquad (22.b)$$

$$r_{1c}(t)=\frac{B'_{mc}(D)}{P_{2c}(D)R_{1c}(D)}r_c(t) \qquad (22.c)$$

with $B'_{mc}(D)$ being the free- design zeros of $W_{mc}(D)$ (i. e. , those of $W_{mc}(D)$ excluding the factor $\overline{Q}_c(D)P_{1c}(D)$). □

The proof is omitted by space reasons. Note that Theorem 2 applies the same philosophy for pole-placement for the continuous reference model as the previously used for the discrete one in Corollary 1 since the coupling signals from the discrete subsystems to the continuous one are cancelled by the controller (14)-(17) with the compensators and compensating signal fulfilling (20)-(22) while the compensating signal in (22.c) is used to cancel the unsuitable plant zeros . A more general choice of r_{1c} (t) based on an arbitrary design of $G_{ic}(D, q)$ (i = 3 , 4) could be established in the same way as addressed in Theorem 1 for the discrete model , although at the expense of more involved calculations .

3. 4_Summary of the controller synthesis method and guidelines for particular designs of interest_

The synthesis of the hybrid controller for the hybrid plant (1) consists of firstly defining the discrete and continuous reference models $W_{md}(q) = B_{md}(q) / A_{md}(q)$ and $W_{mc}(q) = B_{mc}(q) / A_{mc}(q)$ for uniformly bounded reference inputs $r_d[k]$ and $r_c(t)$, $t \in [kT, (k+1)T)$ and all integer $k \geq 0$. Then, u[k] and u (t) , $t \in [kT, (k+1)T)$ and all integer $k \geq 0$ are generated from (6) , with the compensators designed according to Theorem 1 or Corollary 1 , and (15)-(18) with the compensators designed according to Theorem 2, respectively. Particular designs of practical interest are:

Design 1 (*Continuous - time reference model*). The reference input to the continuous-time reference model $W_{mc}(D)$ is piecewise continuous with discontinuities at sampling instants and being constant inbetween sampling instants and the discrete-time reference model $W_{md}(q)$ is the z - transform of $W_{mc}(D)$. Choose the reference signal as $r(t) = r_c(t) = r_c[k] = r_d[k] = r[k]$, $t \in [kT, (k+1)T)$. Thus, the reference output is generated by a unique reference model for all $t \geq 0$. The , in general discontinuous , plant input is generated from (6) and Theorem 1 , or Corollary 1 , for t = k T and from (15) -(18)and Theorem 2 for $t \neq kT$. *The main difference of Design 1 with respect to Design 2 below is that the plant input is generated at sampling instants from a discrete-time model - following philosophy while it is generated from a continuous-time model-matching philosophy inbetween sampling instants despite that a unique continuous - time model is available together with its discretization at sampling instantse.* In other words, the diophantine equation solving the pole-placement problem at sampling instants is of a discrete nature and it is related to the q-operator while that used for the continuous dynamics pole-placement is of a continuous nature and it is related to the D-operator.

Design 2 (*Continuous- time reference model with the controller using periodic plant reparametrization*). $W_{mc}(D)$ is used as the unique reference model at all time. The use of a discrete-time reference model $W_{md}(q)$ is omitted in this design . At each new sampling instant t = kT , the continuous-time description of the plant is reparametrized with the replacement $P_c(D)\ Q_d(q) \rightarrow P_c(D)\ Q_d(q) + Q_c(D) N^u_{cd}(D,q)$ in (3), according to Remark 1, since the right - hand -side terms of (1.a) and (1.c) that involve to u (t) and u [k] have to be summed up when t = k T . Thus , (15) -(18)and Theorem 2 are used to generate the control signal for each t = k T with $r(t) = r[k] = r_c[k]$. Subsequently, $r_{1c}(t) = r_1(t)$ and $r(t) = r_c(t)$, $t \in (kT, (k+1)T)$ and the plant input u (t) is generated from (17) and Theorem 2 for $t \neq kT$. *The main difference of Design 2 with respect to Design 1 is that now the plant input is always generated from Theorem 2 (i. e., from the continuous-time dynamics) with the plant involving a reparametrization at sampling instants* (see Remark 1). The associated pole-placement problem problem is given by two diophantine equations "at" and "inbetween" sampling instants.

Design 3 (*Discrete - time reference model*). The plant input is restricted to be piecewise continuous with discontinuities at sampling instants only while being constant inbetween sampling instants , i. e ., it is generated by a zero-order-hold and u (t) = u [k] = u (kT) , $t \in (kT, (k+1)T)$. Thus , only the discrete-time reference model $W_{md}(q)$ is used in this particular design . Thus, $r_1[k] = r_{1d}[k]$ and $r(t) = r[k] = r_d[k]$. Simple calculus yields $\omega[k] = \ell u[k]$ with $\ell = (\int_0^T e^{A_c(T-t)} d\tau) b_c$, which substituted in (2) and (6) yields directly :

$$Q_d y[k] = \bar{P}_d(q) u[k] \tag{23.a}$$

$$u(t) = u[k] = C^d_{yu}(q) y[k] + C^d_{r_1u}(q) r_{1d}[k] \tag{23.b}$$

all $t \in [kT, (k+1)T)$ with

$$\bar{G}_{1d}(q) = G_{1d}(q) + G^T_{3d}(q)\ell = \bar{G}'_{1d}(q) P_{2d}(q) \tag{23.c}$$

$$\bar{P}_d(q) = P_d(q) + Q_d(q) c^T \ell \tag{23.d}$$

$$C^d_{yu}(q) = \frac{G_{2d}(q)}{L_d(q) - \bar{G}_{1d}(q)} \tag{23.e}$$

$$C^d_{r_1u}(q) = \frac{R_{1d}(q)}{L_d(q) - \bar{G}_{1d}(q)} \tag{23.f}$$

the model- matching problem is solved by applying the controller to the plant by recombining eqns. 26.a while solving the diophantine equation (8) with the replacement $G_{1d} \to \overline{G}_{1d}$ in the solution polynomials $\overline{G}'_{1d}$ and G_{2d} , which are unique if $\deg(\overline{G}'_{1d}) = \deg(P_{1d}) - 1$ and the compensating signal $r_1(t) = r_1[k] = r_{1d}[k] = R_{1d}^{-1} P_{2d}^{-1} B'_{md} r[k]$, all $t \in [kT, (k+1)T]$ and all integer $k \geq 0$.

Design 4 (*General combined continuous - time and discrete-time reference models with large sampling periods*). This design keeps both Objectives 1-2. The discrete-time reference model $W_{md}(q)$ is designed with a large sampling period compared to the dominant constant of the continuous- time subsystem while keeping Assumption 3. I this context, Objective 2 over the continuous- time reference model $W_{mc}(D)$ is the basis of the overall design . Objective 1 is used for periodic testing of the closed-loop performance and eventual re- adjustment of the continuous-time model in case of performance' s test failure. If such a test fails in terms of excessive deviations of the sampled output from its neighbouring values generated by Objective 1 then either the high-frequency gain of $W_{mc}(D)$ or its reference input $r_c(t)$ can be re-updated appropriately. This model re-updating procedure makes justifiable the use of two separate continuous-time and discrete- time reference models and two associated control objectives as stated in the general design procedure.

4. Concluding Rermarks

This paper has dealt with the model - following design of a class of single- input single- output linear hybrid systems which consists of a continuous-time subsystem and a digital one which are coupled in general . The design philosophy is the simultaneous use of a continuous - time model to be matched inbetween sampling instants as well as a discrete- time one which has to be matched at sampling instants .

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References:

[1] CLARKE, D. W. , 1985, " Introduction to self- tuning controllers " in *Self - Tuning and Adaptive Control : Theory and Applications* . IEE Control Engineering Series (Revised 2nd edition , C. J. Harris and S. A. Billings Eds.) , Peter Peregrinus Ltd., Chapter 2 , pp. 36 - 70 (London : Peter Peregrinus).

[2] KABAMBA, P. T. , and HARA , S. , 1994, " Worst-case analysis and design of sampled - data control systems ", *IEEE Trans. Automat . Control* , Vol. AC - 38 , No. 9 , pp. 1337- 1357.

[3] DE LA SEN , M. , 1996 , "On the reachability and observability of hybrid multirate sampling systems " , *Computers and Mathematics with Applications* , Vol. 31 , No. 1 , pp. 109 - 122.

[4] DE LA SEN, M. , 2006 , "On the adaptive control of a class of SISO dynamic hybrid systems”, *Applied Numerical Mathematics,*Vol. 56, No. 5, pp. 618-647, 2006.

[5] DE LA SEN , M. and LUO , N. S., 1996 , " A note about total stability of a class of hybrid systems ", *Informatica* , Vol. 17, No. 4 , pp. 565-576, 2006 .

[6] DORATO, P. , FORTUNA , L. and MUSCATO , G. , 1992 , *Robust Control for Unstructured Perturbations. An Introduction* (Lecture Notes in Control and Information seies , 168 , M. Thoma and A. Wyner eds.) (Berlin : Springer Verlag).

[7] CHALAM, V. V., 1987 , *Adaptive Control Systems : Techniques and Applications* (Electrical Engineeering and Electronics Series / 39) (New York : Marcel Dekker).

[8] KAILATH , T. , 1980 , Linear Systems (Englewood Cliffs , N. J. : Prentice - Hall).

[9] NARENDRA , K. S. , KHALIFA , I. H. , and ANNASWAMY, A. M. , 1985 , " Error models for stable hybrid adaptive systems " , *IEEE Transactions on Automatic Control* , Vol. AC- 25 , No. 3 , pp. 339 - 347.

[10] SASTRY, S. and BODSON , M. , 1989 , *Adaptive Control . Stability , Convergence and Robustness* , Prentice - Hall Advanced Referenced Series Engineering, T. Kailath Ed. (Englewood Cliffs:Prentice - Hall).

[11] VOULGARIS, P.G. and BAMIEH , B. , 1993 , "Optimal H^{∞} and H^2 control of hybrid multirate systems " , *Systems and Control Letters* , Vol. 20 , pp. 249 - 261.

[12] DE LA SEN, M., “Adaptive control of single-input single-output hybrid systems possessing interacting discrete and continuous-time dynamics*”, Discrete Dynamics in Nature and Society*, Vol. 2005, no. 3, pp. 299-329, 2005.

[13] DE LA SEN, M. and ALASTRUEY, C.F., “ Stability results for two classes of linear time-delay and hybrid systems, *Engineering Computations*, Vol. 21, Nos. 7-8, pp. 718-735, 2004.