On Lagrange multivariate interpolation problem in generalized degree polynomial spaces

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Abstract: The aim of this paper is to study the Lagrange multivariate interpolation problems in the space of polynomials of w-degree n. Some new results concerning the polynomial spaces of w-degree n are given. An algorithm for obtaining the w-minimal interpolation space is presented.

Key–Words: Lagrange multivariate polynomial interpolation, w- homogeneous polynomial spaces, Generalize degree.

1 Introduction

The Lagrange interpolation is the most frequently used type of interpolation. It supposes an approximation of an unknown function using only the values of this function on a set of points. The interpolation function matches the values of initial function on the set of given points. Let be $\Theta = \{\theta_1, \ldots, \theta_n\} \subset R^d$ a set of arbitrary points and $\mathcal{F} \supset \Pi^d$ a set of functions which includes polynomials. The Lagrange interpolation problem is to find a polynomial subspace $\mathcal{P}$ such that for an arbitrary function $f \in \mathcal{F}$ there exists an unique polynomial $p \in \mathcal{P}$ such that

$$f(\theta_i) = p(\theta_i),\ \forall\, i \in \{1, \ldots, n\} \tag{1}$$

In this case the pair $(\Theta, \mathcal{P})$ is called correct. In [11] is proved that the measure of the set of knots for which the Lagrange interpolation has not unique solution is zero and the Lagrange interpolation is almost always possible. In [1], C. de Boor proved that the correctness of the pair $(\Theta, \mathcal{P})$ is equivalent with the equalities:

$$dim\, \mathcal{P} = \#(\Theta) = dim\, \mathcal{P}|_\Theta, \tag{2}$$

with $\mathcal{P}|_\Theta = \{f|_\Theta \mid f \in \mathcal{P}\}$.

One of the difficulty encountered by the multidimensional Lagrange interpolation is the so called "loss of Hair": for every finite dimensional linear space V of continuous functions on R^d,$d > 1$ there are sets of points $\Theta \subset R^d$, such that $dim\ V = \#(\Theta) > dim\ V|_\Theta$. More, for a given n and $d > 1$ there is not a n-dimensional correct polynomial space for every sets of points having cardinality n.

Another difficulty is given by the fact that the set of dimensions of polynomials in d variables does not cover the entire set of natural numbers:

$$dim\, \Pi_n^d = \begin{pmatrix} n+d \\ d \end{pmatrix} \tag{3}$$

More, there are many polynomials in d variables, linear independent, with the same total degree.

There are many studies of Lagrange multivariate interpolation for rectangular and triangular grids of points. For these configurations the tensor product method is successfully used (see [9]). Other authors look for set of points Θ, such that the pair (Θ, Π_n^d) be correct (see [8], [10], [11]). These methods are not working when the points are given and can not be modified in the interpolation process.

C. de Boor and A. Ron found a minimal interpolation space for a given set of points $\Theta \in R^d$ (see [1]). They called this space Π_Θ:

$$\Pi_\Theta = (Exp_\Theta)\!\downarrow = span\{g\!\downarrow;\ g \in Exp_\Theta\} \tag{4}$$

$$Exp_\Theta = span\{e_\theta;\ \theta \in \Theta\}, \tag{5}$$

The Lagrange interpolation is an ideal interpolation scheme be cause

$$ker\ \Theta = \{p \in \Pi | p(\theta) = 0, \forall\, \theta \in \Theta\} = I_\Theta$$

is a polynomial ideal.

Multivariate ideal interpolation schemes are deeply connected with H-bases. Any ideal interpolation space with respect to a set of conditions Λ, can be obtained like a space of reduced polynomials modulo a H-basis of the ideal $ker(\Lambda)$. The definition of a H-basis depends of the notion of degree used in the grading decomposition of the polynomial spaces and of the inner product used in the reduction process.

We studied,in [21], in the case of bivariate polynomials, a generalized degree, introduced by T. Sauer and named w-degree.

The aim of this article is to study the Lagrange interpolation problem in the space of polynomials of w-degree, by using the connection between interpolation and reduction modulo a H-basis of the ideal $ker\ \Theta$. We want to find, in the space of polynomials of w-degree a minimal interpolation space for the conditions Θ.

In order to do this, in section 2 we present some notions and results related to the w-degree and in section 3 we present the main results of this article. Conclusions are given in section 4.

2 The w-degree grading

In [17], T. Sauer introduced the w-degree of a monomial:

Definition 1 *The w-degree of the monomial x^α is*

$$\delta_w(x^\alpha) = w \cdot \alpha = \sum_{i=1}^{d} w_i \cdot \alpha_i,$$

$\forall\ \alpha \in N^d,\ w = (w_1, \ldots, w_d) \in N^d,\ x \in R^d.$

This degree induces on the space of polynomials in d variable a grading, in the sense given in [19].

A Γ-grading is defined as follows.

Let $(\Gamma, +)$ denotes an orderer monoid, with respect to the total ordering $\prec$, such that: $\alpha \prec \beta \Rightarrow \gamma + \alpha \prec \gamma + \beta,\ \forall\ \alpha, \beta, \gamma \in \Gamma$.

Definition 2 *([19]) A direct sum*

$$\Pi = \bigoplus_{\gamma \in \Gamma} \mathcal{P}_\gamma^{(\Gamma)} \qquad (6)$$

is called a grading induced by Γ, or a Γ-grading, if $\forall\ \alpha,\ \beta \in \Gamma$

$$f \in \mathcal{P}_\alpha^{(\Gamma)},\ g \in \mathcal{P}_\beta^{(\Gamma)} \Rightarrow f \cdot g \in \mathcal{P}_{\alpha+\beta}^{(\Gamma)} \qquad (7)$$

The total ordering induced by Γ gives the notion of degree for the components in $\mathcal{P}_\gamma^{(\Gamma)}$. Each polynomial $f \neq 0$, has a unique representation

$$f = \sum_{i=1}^{s} f_{\gamma_i},\ f_{\gamma_i} \in \mathcal{P}_{\gamma_i}^{(\Gamma)}; \quad f_{\gamma_i} \neq 0 \qquad (8)$$

The terms f_{γ_i} represent the Γ- homogeneous terms of degree γ_i.

Assuming that $\gamma_1 \prec \ldots \prec \gamma_s$, the Γ- homogeneous term f_{γ_s} is called the leading term or the maximal part of f, denoted by $f^{(\Gamma)} \uparrow$.

In order to prove that the w-degree induces a grading, we introduce the followings sets:

$$A_{n,w}^0 = \{\alpha \in N^d \mid w \cdot \alpha = n\},\ w \in (N^*)^d,\ n \in N \qquad (9)$$

$$r_{n,w} = \#(A_{n,w}^0) \qquad (10)$$

$$N_A = \{n \in N \mid \exists \alpha \in A_{n,w}^0\} \qquad (11)$$

Proposition 1 *The direct sum*

$$\Pi = \bigoplus_{\gamma \in N_A} \mathcal{P}_\gamma^{(N_A)} \qquad (12)$$

is a N_A-grading, in the sense given by the definition 2

Proof: The polynomial homogeneous subspace of w-degree n can be rewritten as:

$$\Pi_{n,w}^0 = \left\{ \sum_{\alpha \in A_{n,w}^0} c_\alpha x^\alpha \mid c_\alpha \in R,\ \alpha \in N^d \right\}$$

Let be $\alpha \in A_{n,w}^0$ and $\beta \in A_{m,w}^0$. The w- degree of the monomial $x^\alpha \cdot x^\beta$ is :

$$\delta_w(x^\alpha \cdot x^\beta) = w \cdot (\alpha + \beta) = (w \cdot \alpha) + (w \cdot \beta)$$

$$\Leftrightarrow \delta_w(x^\alpha \cdot x^\beta) = \delta_w(x^\alpha) + \delta_w(x^\beta)$$

This relation proves that if $f \in \Pi_{n,w}^0$ and $g \in \Pi_{m,w}^0$, then $f \cdot g \in \Pi_{n+m,w}^0$. □

We will name, w-degree grading, the grading obtained by choosing in definition 2, $\Gamma = N_A$ with the natural total ordering. The leading term corresponding to this grading will be denoted by $f \uparrow_w$

3 The interpolation problem

Let be

$$\Theta = \{\theta_i \mid \theta_i \in R^d,\ i = 1, \ldots, n\}$$

We want to obtain a space of minimal w-degree for the conditions Θ.

We will use the following theorem:

Theorem 1 *([19]) Let be $\Lambda \subset \Pi'$ an ideal interpolation scheme and $\mathcal{H}$ a H-basis for the ideal $ker\ \Lambda$. Then, the set of reduced polynomials modulo a $\mathcal{H}$, $\mathcal{P}_\mathcal{H} = \Pi_{\rightarrow \mathcal{H}}$, is a minimal interpolation space for the conditions Θ and the interpolation operator is the reducing operator modulo $\mathcal{H}$ - basis, that is*

$$L_{\mathcal{P}_\mathcal{H}}(q) = q \rightarrow_\mathcal{H};\ q \in \Pi. \qquad (13)$$

For the reducing process modulo a H-basis we will use the inner product

$$< f, p >= (p(D)f)(0) = \sum_{\alpha \in N^d} \frac{D^\alpha p(0) D^\alpha f(0)}{\alpha!}, \tag{14}$$

$f,\ p \in \Pi^d$.

In our problem, the reducing operator modulo a $\mathcal{H}$ - basis $(p_1, \dots, p_m)$, associates to each polynomial p, its reduced part,r, related to the w-grading. Every polynomial p has an unique representation:

$$p = \sum_{k=1}^{m} q_k \cdot p_k + r, \text{ with} \tag{15}$$

$$\delta_w(q_k) + \delta_w(p_k) \leq \delta_w(p) \text{ and}$$

$$r = \sum_{n=0}^{\delta_w(p)} r_n, \text{ with} \tag{16}$$

$$r_n \perp V_n(p_1, \dots, p_m). \tag{17}$$

The orthogonalisation is made related to a given inner product. In fact, the following algorithm is used in order to calculate the reduced polynomial :

Reduction algorithm

For $n = \delta_w(p), \dots, 0$

For $j = 1, \dots, m$

we calculate the polynomials $q_j^n \in W_n(p_1, \dots, p_j)$, given by

$$q_j^n = \sum_{k=1}^{j} q_{j,k}^n \cdot p_k \uparrow,\ q_{j,k}^n \in \Pi^0_{n-\delta_w(p_k)} \tag{18}$$

such that

$$f_n \uparrow - q_j^n \perp W_n(p_1, \dots, p_j), \tag{19}$$

where, first $f_{\delta_w(p)} = p$.

We obtain the component

$$r_n = f_n \uparrow - \sum_{j=1}^{m} q_j^n = f_n \uparrow - \sum_{j=1}^{m} \sum_{k=1}^{j} q_{j,k}^n \cdot p_k \uparrow, \tag{20}$$

with $r_n \perp V_n(p_1, \dots, p_m)$.

The inductive reduction process is realized for every w-homogeneous components of p, by choosing

$$f_{n-1} = f_n - r_n - \sum_{j=1}^{m} \sum_{k=1}^{j} q_{j,k}^n \cdot p_k \tag{21}$$

The main result of this section is given by the following theorem:

Theorem 2 *Let be* $\Theta = \{\theta_i \mid \theta_i \in R^d,\ i = 1, \dots, n\}$, $\mathcal{H}$ *a H-basis for the ideal* $I = ker\ \Lambda$, *with respect to the inner product given in (14) and the* w*-grading, then*

$$\Pi_{\rightarrow \mathcal{H}} = \bigcap_{q \in ker\ \Theta} ker\ q \uparrow_w (D) = (\Pi_\Theta)_w, \tag{22}$$

with

$$(\Pi_\Theta)_w = (Exp_\Theta)\downarrow_w = span\{g\downarrow_w;\ g \in Exp_\Theta\} \tag{23}$$

Proof:$(\Pi_\Theta)_w$ is a w- minimal interpolation space for Θ. Using a proof as in [17] we obtain that

$$\bigcap_{q \in ker\ \Theta} ker\ q \uparrow_w (D) = (\Pi_\Theta)_w.$$

The first equality results from theorem 1. □

4 Conclusions

In this paper we discuss a Lagrange interpolation scheme in the space of polynomials of w-degree. This type of interpolation schemes appears in the practical problems in which the coordinates are dependent of time and the condition of minimal degree of the interpolation polynomial is replaced with the condition of w-minimal degree of the interpolation polynomial. Be cause Lagrange interpolation scheme are ideal interpolation schemes, we can use the connection between the interpolation and the reduction process modulo a H-basis of the ideal $ker\ \Theta$. We found a form of the interpolation polynomial in the space of polynomials of w-degree. If we can construct the interpolation space, we know the space of reduced polynomials. On the other hand, we can apply the algorithm for obtaining the reduced polynomials modulo a H-basis of the ideal $ker\ \Theta$, in order to obtain the interpolation space.

The construction of the w-minimal interpolation space for some practical problems, the interpretation of the results and the comparison with the classical minimal interpolation space are our further directions of study.

Acknowledgement: This work benefits from founding from the research grant of the Romanian Ministry of Education and Research, cod CNCSIS 33/2007.

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