Ocean Waves – A Preliminary Study

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*Abstract :*Ocean energy conversion has been of interest for many years. Recent developments such as concern over global warming have contributed in renewing the interest in this topic. The present work seeks the maximization of wave energy conversion. With this purpose, *Small Amplitude Wave Theory* was developed since the income wave power plant is determined when amplitudes are low, i.e., when linear theory is applicable. Wave-energy transport expression was maximized and a harmonic wave was found for a typical water depth of *50 meters*. Our goal is to maximize wave energy on the entrance of Wave Energy Converters, to increase the conversion of the energy available and simultaneously to regularize the wave flow.

Key-Words: Ocean renewable energy, Wave energy conversion, Optimization

1 Introduction

The ocean holds a tremendous amount of untapped energy. Although the oil crisis of the 1970s increased interest in ocean energy, relatively few people have heard of it as available energy alternative. In fact, hydroelectric dams are the only well known, mass producing water-based energy, but the ocean is also a highly exploitable water-based energy source. Ocean energy comes in a variety of forms such as geothermal vents, and ocean currents and waves. The most commercially viable resources studied so far are ocean currents and waves which have both undergone limited commercial development. Estimates conclude that there is approximately 8,000-80,000 TWh/yr or 1-10 TW of wave energy in the entire ocean [8], and on average, each wave crest transmits 10-50 kW/m.

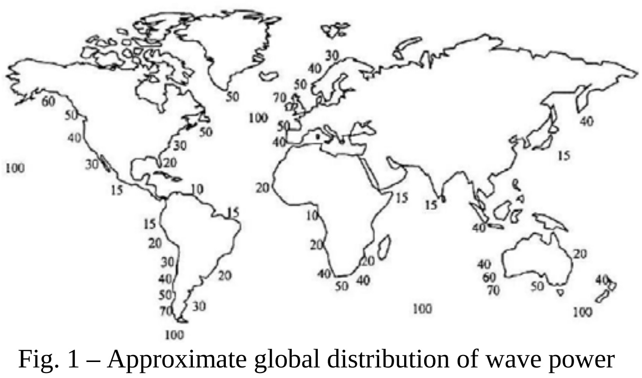

Fig. 1 – Approximate global distribution of wave power levels (kW/m of wave front) *[8]*

The present work seeks the maximization of wave energy conversion. With this purpose, *Small Amplitude Wave Theory* was developed based on this quote *[3]*:

"…for a wave power plant the income is determined by the annual energy production, which is essentially during most times of the year, when amplitudes are low, that is, when linear theory is applicable."

2 Problem Formulation

In an *Eulerian* system of coordinates, a surface wave problem generally involves three unknowns *[5]*:

- The fluid particle velocity;
- The hydrodynamic pressure (generally known at the free surface);
- And the free surface elevation (or total water depth).

Since a general method of solution is impossible, a number of simplifying assumptions are made to apply to particular cases *[5]*.

However, instead of dealing with these unknowns directly, it is more convenient to relate them to more accessible typical values. Three typical values are used:

1. A typical value of the free surface elevation – *Wave height, H*;
2. A typical value of the horizontal length – *Wavelength, λ;*
3. *The wave depth, h*

Although the relationships between the convective terms and these three typical values are not simple, their relative values are of considerable help in classifying the water wave theories. As the free surface elevation decreases, the particle velocity also decreases. Thus, when the wave height, H, tends to zero, the convective term, which is related to the square of the particle velocity, is an infinitesimal *[5]*. Consequently, the convective terms can be neglected and the theory can be linearized.

Considering three relations between typical values:

$$\frac{H}{\lambda}\ ,\ \frac{H}{h}\ ,\text{e}\ \frac{h}{\lambda}.$$

The relative importance of the convective terms increases as the values of these three relations increase.

In deep water (small H / h and small h / λ), the most significant relation is H / λ , which is called the *wave steepness*. In shallow water, the most significant relation is H/h, which is called the *relative height*. The relation h / λ is called relative depth *[5]*.

2.1 Mathematical Development

Waves can mathematically be described through several parameters. The most important are: *wavelength*, λ, *wave height*, H, and *water depth*, h, where waves spread.

Table 1 and *Fig. 2* summarize a commonly used wave energy nomenclature *[8]* that is also used here.

Table 1 - Wave Nomenclature *[8]*
(Illustration: Fig. 2)

SWL	mean seawater level (surface)
h	depth below SWL [m]
H	wave height [m]
a	wave amplitude [m]
T	wave period [s]
λ	wavelenght [m]
k	wave number
ω	wave frequency [rad/s]
c	celerity (phase velocity) [m/s]
c_g	group velocity [m/s]
E	wave stored energy [J/m^2]
J	wave-energy transport [kW/m]
ρ	water density [1020 kg/m^3]
g	gravitational constant [9,81 m/s^2]

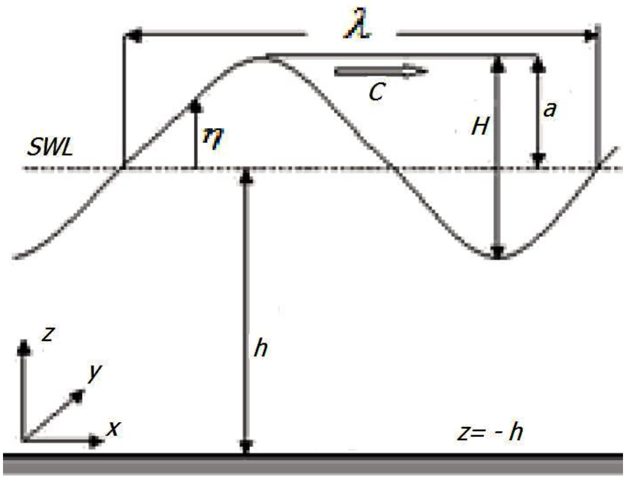

Fig. 2 – Waves nomenclature *[8]*.

2.1.1 Small Amplitude Wave Theory

The wave elevation is given by *[4]:*

$$\eta(x,t) = \frac{H}{2}\cos(k\,x - \omega\,t) \tag{1}$$

A useful relation to be considered is *dispersion relation [3]*:

$$\omega^2 = g\,k\,\tanh(k\,h) \tag{2}$$

Two approximations are especially useful:

1. *Deep water ($h>>\lambda$)*, so, $kh>>1$, as a result, $\tanh(kh)\approx 1$.
2. *Shallow water ($h<<\lambda$)*, so, $kh<<1$, as a result, $\tanh(kh)\approx kh$.

For these two limits the dispersion relation is given by:

1. *Deep water*: $\omega^2 = g\,k$ (3)
2. *Shallow water:* $\omega^2 = g\,k^2\,h$ (4)

Deep Water

Depending on the desired accuracy, it may be assumed that deep water case is when the depth, h, is one half of the wavelength *[3]*.

Wavelength is given by:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\,g}{\omega^2} = \left(\frac{g}{2\pi}\right)T^2 \tag{5}$$

Note: Shallow water case is not considered on this work.

Phase Velocity or Celerity

For constant water depth, h, the phase velocity is, in the general case,

$$c = \frac{\omega}{k} \tag{6}$$

Dispersion relation gives,

$$c = \frac{1}{k}\sqrt{g\,k\,\tanh(k\,h)} = \sqrt{\frac{g\,\tanh(k\,h)}{k}} \tag{7}$$

Dispersion relation approximation for deep water case gives:

$$c = \sqrt{\frac{g}{2\pi}\,\lambda} \tag{8}$$

In deep water, the phase speed depends on wave length or wave frequency. Longer waves travel faster. Thus, deep-water waves are said to be dispersive *[3]*.

Group Velocity

The definition of group velocity in two dimensional flow is *[3]*:

$$c_g = \frac{\partial \omega}{\partial k} \tag{9}$$

Hence,

$$c_g = \frac{c}{2}\left(1 + \frac{2\,k\,h}{\sinh(2\,k\,h)}\right) \tag{10}$$

Dispersion relation approximation for deep water case gives:

$$c_g = \frac{g}{2\varpi} = \frac{c}{2} \tag{11}$$

Wave Stored Energy

The total stored energy in a wave is the sum of potential energy *(E_P)* and kinetic energy *(E_C)*. For a regular wave the time-average stored energy per unit (horizontal) area is *[3]:*

$$E = E_P + E_C = \frac{\rho\,g\,H^2}{8} \tag{12}$$

Wave-Energy Transport

Wave-energy transport is the power per unit width of wave front – *J*. It's given by *[3]*:

$$J = \frac{\rho\,g^2\,D(kh)}{4\,\omega}\left(\frac{H}{2}\right)^2, \tag{13}$$

where *D(kh)* is called *"depth function"* and it's given by:

$$D(kh) = \tanh(k\,h)\cdot\left(1 + \frac{2\,k\,h}{\sinh(2\,k\,h)}\right) \tag{14}$$

For deep water case we have:

- $\omega = \sqrt{g\,k} = \sqrt{\dfrac{2\pi\,g}{\lambda}}$;
- $D(kh) \approx 1$.

Thus,

$$J = \frac{\rho\,g^2\,H^2}{16}\sqrt{\frac{\lambda}{2\pi\,g}} = \frac{\rho\,H^2}{16}\sqrt{\frac{g^3\,\lambda}{2\pi}} \tag{15}$$

By separating the variables we get the final expression of *Wave-Energy Transport*:

$$J = \frac{\rho}{16}\sqrt{\frac{g^3}{2\pi}}\,H^2\,\sqrt{\lambda} \tag{16}$$

2.1.1 Range of Applicability of the model [6]

The limits to small amplitude wave theory are shown in *Fig. 3*:

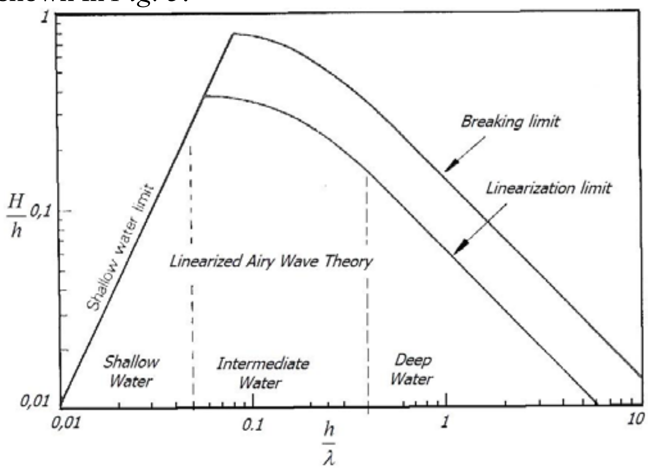

Fig. 3 – Range of applicability of Small Amplitude Wave Theory *[6]*.

This figure also shows the limit of wave steepness above which the waves would "break":

$$\left(\frac{H}{\lambda}\right)_{quebra} = 0{,}140\,\tanh(k\,h) \tag{17}$$

However, before this breaking limit, the neglected non-linear terms become significant. Thus, a new limit becomes:

$$\frac{H}{\lambda} << \frac{2}{\pi}\tanh(k\,h) \tag{18}$$

If the error is limited to just below 10% then a convenient limit becomes:

$$\frac{H}{\lambda} \le \frac{1}{16}\tanh(k\,h) \tag{19}$$

If we consider deep water $\left(h/\lambda > \frac{1}{2}\right)$, the limit becomes:

$$\frac{H}{\lambda} \le \frac{1}{16} \tag{20}$$

3 Maximization of Wave-Energy Transport

Given the objective function:

$$J = \frac{\rho}{16}\sqrt{\frac{g^3}{2\pi}}\, H^2 \sqrt{\lambda}$$

By differentiating with respect to variables *H* and *λ*, considering $K = \frac{\rho}{16}\sqrt{\frac{g^3}{2\pi}}$ as a constant value we obtain:

$$\frac{\partial J}{\partial H} = 2K\, H \sqrt{\lambda}$$

$$\frac{\partial J}{\partial \lambda} = \frac{1}{2} K\, H^2 \frac{1}{\sqrt{\lambda}}$$

It can be concluded from partial derivatives that there aren't relative extremes.

The maximum value will be given by imposing physical and mathematical restrictions:

1. Deep water case: $\frac{h}{\lambda} > \frac{1}{2}$;
2. Deep water linearization limit: $\frac{H}{\lambda} \leq \frac{1}{16}$;

Since the objective function is a growing function, its absolute maximum is given by the relative's extremes of considered intervals, we obtain:

1. $\lambda = 2h$;
2. $H = \frac{\lambda}{16} \Leftrightarrow H = \frac{h}{8}$.

Substituting on wave-energy transport expression, it becomes:

$$J = \frac{\rho}{16}\sqrt{\frac{g^3}{2\pi}}\, H^2 \sqrt{\lambda}$$

$$J = \frac{\rho}{16}\sqrt{\frac{g^3}{2\pi}} \left(\frac{h}{8}\right)^2 \sqrt{2h}$$

$$J = \frac{\rho}{1024}\sqrt{\frac{g^3}{\pi}}\, h^{5/2} \qquad (21)$$

Considering $\rho = 1020\, kg/m^3$ for sea water, the maximum value for wave-energy transport we can obtain with small amplitude wave theory with a typical water depth of *50 m [2]* is:

$$J = \frac{1020}{1024}\sqrt{\frac{9{,}81^3}{\pi}}\, 50^{5/2}$$

$$J \approx 305\, kW/m$$

With this considered water depth, we obtain for the wave height and wavelength:

$$\lambda = 100\ m$$

$$H = 6{,}25\ m$$

The wave which transports maximum energy (on water with *50 m* of depth) is given by expression:

$$\eta(x,t) = \frac{H}{2}\cos(k\,x - \omega\,t)$$

where wave number, *k*, is given by $2\pi/\lambda$ and angular frequency, *ω*, is given by dispersion relation approximation for deep water case *(3)*:

$$\eta(x,t) = \frac{H}{2}\cos(k\,x - \omega t)$$

$$\eta(x,t) = \frac{6{,}25}{2}\cos\left(\frac{2\pi}{100}\,x - \sqrt{\frac{9{,}81\times 2\pi}{100}}\,t\right)$$

$$\eta(x,t) = 3{,}125\cos\left(0{,}02\pi\, x - \sqrt{0{,}1962\pi}\, t\right)$$

4 Conclusion

>From this preliminary work the estimated maximum value for wave-energy transport is approximately *305 kW/m* of wave frontage on waters with *50 m* depth. The attained results neglect the dissipative terms from the interaction between the wind and the waves and between the waves and the seabed. On future work it is intended to include these terms and validate the values with the help of a *Gough-Stewart Platform* that is used to simulate the water inertia.

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