On Thermic Exposure of Piezo-Thermoelastic Plates

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Abstract: We consider a plate infinite in extent, bounded by two parallel planes, filled by a heat-conducting piezoelectric material with hexagonal symmetry. It is completely coated by electrodes which are infinitesimally thin, so that all their mechanical effects may be ignored. On the lower face the displacement is prescribed, as when e.g. the panel is welded above a fixed rigid body, and on the upper face the stress vanishes. We study quasi-static processes controlled by a thermic exposure on the upper plane, which varies very slowly with the time, e.g. solar exposure. The difference of electric potential between the two bounding planes is computed in terms of the temperature on the upper face.

Key–Words: Thermo-piezoelastic, Panel, Thermic-exposure, Solar-exposure, Quasi-statics, Piezoelectricity, Ceramics

1 Introduction

The constitutive equations for the mechanical Cauchy stress tensor $\mathbf{t}$, electric displacement vector $\mathbf{D}$, heat flux vector $\mathbf{q}$, and specific entropy η in a linear thermo piezoelastic material write as

$$t_p = c_{pq}\, S_q - e_{ip}\, E_i - \beta_p\, T\,, \tag{1}$$

$$D_i = e_{iq}\, S_q + \epsilon_{ik}\, E_i + \tilde{\omega}_i\, T\,, \tag{2}$$

$$q_i = \kappa_{ij}\, T_{,\,j} + \kappa^E_{ij}\, E_j\,, \tag{3}$$

$$\eta = \eta_0 + \frac{\gamma}{T_0}\, T + \frac{1}{\rho_o}\, (\beta_p\, S_p + \tilde{\omega}_i\, E_i)\,, \tag{4}$$

with $p,\, q = 1,\, 2,\, \dots,\, 6, \quad i,\, j = 1,\, 2,\, 3,$
$E_i = -\frac{\partial\phi}{\partial x_i}$ electric displacement vector,
ϕ electrostatic potential,
T incremental absolute temperature.

The above linear constitutive equations are specified in terms of the material constants $c_{pq} =$ elastic moduli, $e_{ikl} =$ piezoelectric moduli, $\beta_{kl} =$ thermic stress moduli, $\kappa^E_{kl} =$ dielectric susceptibility, $\tilde{\omega}_k =$ pyroelectric polarizability, $\epsilon_{kl} =$ permittivity moduli, $\kappa_{kl} =$ Fourier coefficients, $\gamma =$ heat capacity, $\eta_o =$ entropy at the natural state, $T_o =$ absolute temperature at the natural state, $\rho_o =$ mass-density at the natural state.

The local equations corresponding to the (i) balance law of linear momentum, (ii) Maxwell's equation, and (iii) balance law of conservation of energy, respectively write as

$$t_{kl,\,k} + \rho_o(f_l - \ddot{u}_l) = 0\,,$$
$$D_{k,\,k} = q_e\,, \qquad \rho_o\,\theta\,\dot{\eta} - q_{k,\,k} = \rho_o\, h\,, \tag{5}$$

where

f_l is the body force density,

q_e is the free (or prescribed) body charge density,

h is the heat source per unit mass.

For $f_l = q_e = h = 0$, the above balance equations yield the following equilibrium equations

$$t_{kl,\,k} = 0\,, \qquad D_{k,\,k} = 0\,, \qquad q_{k,\,k} = 0\,. \tag{6}$$

Here we consider a material with hexagonal symmetry in the crystal class $C_{6\,\nu} = 6mm.$. The polarized ferroelectric ceramics have the symmetry of such a class.

Choosing x_3 in the poling direction, the constitutive equations (1), (2) of stress and electric displacement become (see ([1], p.58)

$$\begin{aligned}
t_1 &= c_{11}u_{1,\,1} + c_{12}u_{2,\,2} + c_{13}u_{3,\,3} + e_{31}\varphi_{,\,3} - \beta_1 T\\
t_2 &= c_{12}u_{1,\,1} + c_{11}u_{2,\,2} + c_{13}u_{3,\,3} + e_{31}\varphi_{,\,3} - \beta_2 T\\
t_3 &= c_{13}u_{1,\,1} + c_{13}u_{2,\,2} + c_{33}u_{3,\,3} + e_{33}\varphi_{,\,3} - \beta_3\, T\\
t_4 &= c_{44}\,(u_{3,\,1} + u_{1,\,3}) + e_{15}\,\varphi_{,\,2}\\
t_5 &= c_{44}\,(u_{3,\,1} + u_{1,\,3}) + e_{15}\,\varphi_{,\,1}\\
t_6 &= c_{66}\,(u_{1,\,2} + u_{2,\,1})
\end{aligned} \tag{7}$$

$$D_1 = e_{15}u_{3,1} + e_{15}u_{1,3} - \epsilon_{11}\varphi_{,1} + \tilde{\omega}_1 T$$
$$D_2 = e_{15}\left(u_{3,2} + u_{2,3}\right) - \epsilon_{11}\varphi_{,2} + \tilde{\omega}_2 T$$
$$D_3 = e_{31}u_{1,1} + e_{31}u_{2,2} + e_{33}\, u_{3,3} - \epsilon_{33}\varphi_{,3} + \tilde{\omega}_3 T$$

Hence the equilibrium field equations for a body formed with such a material write as

$$\begin{aligned} -\beta_{21}\, T_{,2} \,+\, c_{11}u_{1,11} \,+\, (c_{12}+c_{66})u_{2,12} \\ +\, (c_{13}+c_{44})u_{3,13} \,+\, c_{66}u_{1,22} \,+\, c_{44}u_{1,33} \\ +\, (e_{31}+e_{15})\varphi_{,13} \,=\, 0 \end{aligned} \quad (8)$$

$$\begin{aligned} -\beta_{22}\, T_{,2} \,+\, c_{66}u_{2,11} \,+\, (c_{66}+c_{12})u_{1,12} \\ +\, c_{11}u_{2,22} \,+\, (c_{13}+c_{44})u_{3,23} \,+\, c_{44}u_{2,33} \\ +\, (e_{31}+e_{15})\varphi_{,23} \,=\, 0 \end{aligned} \quad (9)$$

$$\begin{aligned} -\beta_{23}\, T_{,2} \,+\, c_{44}u_{3,11} \,+\, (c_{44}+c_{13})u_{1,31} \\ +\, c_{44}u_{3,22} \,+\, (c_{44}+c_{13})u_{2,23} \,+\, c_{33}u_{3,33} \,+ \\ +\, e_{15}\varphi_{,11} \,+\, e_{15}\varphi_{,22} \,+\, e_{33}\varphi_{,33} \,=\, 0 \end{aligned} \quad (10)$$

$$\begin{aligned} e_{15}u_{3,11} \,+\, (e_{15}+e_{31})u_{1,13} \,+ \\ +\, e_{15}u_{3,22} \,+\, (e_{15}+e_{31})u_{2,32} \,+ \\ +\, e_{15}u_{3,22} \,+\, e_{33}u_{3,33} \,+ \\ +\, \tilde{\omega}_l\, T_{,l} \,-\, \epsilon_{11}\varphi_{,11} \,-\, \epsilon_{11}\varphi_{,22} \,-\, \epsilon_{33}\varphi_{,33} \,=\, 0 \end{aligned} \quad (11)$$

$$-\kappa_{ll}\, T_{,ll} \,+\, \kappa^E_{ll}\, \varphi_{,ll} \,=\, 0\,. \quad (12)$$

2 Quasi-Static Problems

Here we consider a plate $\mathcal{P}$, infinite in extent and bounded by two parallel planes $x_1 = \pm h$. The plate is filled by a heat-conducting piezoelectric material with hexagonal symmetry, so that some quartzs are included. The panel is completely coated by infinitesimally thin electrodes, so that all their mechanical effects can be ignored.

In order to consider very slow processes for $\mathcal{P}$ depending on the thermic exposure on its upper side $x_1 = h$, we consider a one-parameter family

$$T = T(x_1,\, \tau),\ \varphi = \varphi(x_1,\, \tau),\ u_k = u_k(x_1,\, \tau), \quad (13)$$

of solutions of the equilibrium equations (8)-(12), which are determined by the boundary data on the two bounding planes. The triple (13) represents a one-parameter family of equilibrium states superposed to the reference state. Having as objective the study of the upper exposure to the sun of $\mathcal{P}$, which is settled to a rigid underlying body, the boundary conditions must include the prescriptions of temperature and of vanishing normal stress on the upper bounding plane and the condition of assigned displacement on the lower bounding plane. Hence we will consider homogeneous τ-dependent boundary conditions such as

- temperature prescribed at $x_1 = h$
- electric potential prescribed (e.g. vanishing) at $x_1 = h$
- displacement prescribed (e.g. vanishing) at $x_1 = -h$
- stress prescribed (e.g., vanishing) at $x_1 = h$
- surface free charge prescribed (e.g. vanishing) at $x_1 = -h$

We assume that the prescriptions on the boundary are functions of a parameter τ which depends slowly on time:

$$\tau = \tau(t), \qquad |\tau'(t)| \quad \text{small}$$

We refer to equations (8)-(12) as the *quasi-static equations* and to a solution (13) of the equations (8)-(12), joined with a specified set of boundary conditions, as a *quasi-static process*.

2.1 The boundary-value problem modeling thermic exposures

2.1.1 Statement of the problem

Referring only to thickness static processes (13), the equilibrium field equations (8)- (12) reduce to

$$c_{11}u_{1,11} \,=\, 0 \quad (14)$$

$$c_{66}u_{2,11} \,=\, 0 \quad (15)$$

$$c_{44}u_{3,11} \,+\, e_{15}\varphi_{,11} \,=\, 0 \quad (16)$$

$$e_{15}u_{3,11} \,+\, \tilde{\omega}_1\, T_{,1} \,-\, \epsilon_{11}\varphi_{,11} \,=\, 0 \quad (17)$$

$$-\kappa_{11}\, T_{,11} \,+\, \kappa^E_{11}\, \varphi_{,11} \,=\, 0\,. \quad (18)$$

Next we formulate the boundary-value problem (briefly *b-v problem*) for $\mathcal{P}$, which model equilibrium under sun exposure:

To find the particular solution (13) of the equilibrium field equations (14)-(18), which satisfies the ten boundary conditions listed below, where $\Theta(\tau),\, \Phi(\tau), \Pi_i(\tau),\, \Upsilon_i(\tau),\, \Delta(\tau),\, \Omega(\tau)$ *are given smooth functions of* τ.

1. $T(h,\, \tau) = \Theta(\tau)$
2. $\varphi(h,\, \tau) = \Phi(\tau)$

3. $t_5(h,\,\tau)=\Pi_3(\tau),\quad t_1(h,\,\tau)=\Pi_1(\tau),$ $t_6(h,\,\tau)=\Pi_2(\tau)$

4. $u_i(-h,\,\tau)=\Upsilon_i(\tau),\ (i=1,\,2,\,3)$

5. $-D_1(-h,\,\tau)\,=\,\Delta(\tau)$

6. $-q_1(-h,\,\tau)\,=\,\Omega(\tau)$

2.1.2 General solution of the problem

The general solution to equations (14)-(18) is given by

$$T(x)=T_1\,e^{ax}+T_2 \tag{19}$$

$$\varphi(x)=KT_1\,e^{ax}+F_1x+F_2 \tag{20}$$

$$u_1(x_1)=U_{11}x_1+U_{12} \tag{21}$$

$$u_2(x_1)=U_{21}x_1+U_{22} \tag{22}$$

$$u_3(x)=-EKT_1\,e^{ax}+U_{31}x+U_{32} \tag{23}$$

where $E:=e_{15}/c_{44},\ K:=\kappa_{11}/\kappa_{11}^E,$ $a:=\tilde{\omega}_1\Big(K(\epsilon_{11}+e_{15}^2/c_{44})\Big)^{-1}$, and

$$T_1,\,T_2,\,F_1,\,F_2,\,U_{\alpha 1},\,U_{\alpha 2}\ \ (\alpha=1,\,2,\,3)$$

are arbitrarily chosen smooth functions of τ.

2.1.3 Temperature and electric potential solution to the b-v problem

We split the problem above by firstly solving the three equations (16)–(18) subject to the six boundary conditions

1. $T(h,\,\tau)=\Theta(\tau)$

2. $\varphi(h,\,\tau)=\Phi(\tau)$

3. $t_5(h,\,\tau)=\Pi_3(\tau)$

4. $u_3(-h,\,\tau)=\Upsilon_3(\tau)$

5. $-D_1(-h,\,\tau)\,=\,\Delta(\tau)$

6. $-q_1(-h,\,\tau)\,=\,\Omega(\tau)$

Note that, by the above equations, the 5-th and 6-th boundary condition above respectively becomes

$$-e_{15}u_{3,\,1}+\epsilon_{11}\varphi_{,\,1}-\tilde{\omega}_1T=-\tilde{\omega}_1T_2-e_{15}U_{31}+\epsilon_{11}F_1 \tag{24}$$

and

$$-\kappa_{11}^EKT_{,\,1}+\kappa_{11}^E\varphi_{,\,1}=\kappa_{11}^EKae^{ax}(1-K)T_1+\kappa_{11}^EF_1 \tag{25}$$

Now by imposing these boundary conditions to the general solution and using the constitutive equations, we find the following relations:

$$T_1\,e^{ah}\,+\,T_2\,=\,\Theta \tag{26}$$

$$K\,T_1\,e^{ah}\,+\,F_1\,h\,+\,F_2\,=\,\Phi \tag{27}$$

$$c_{44}(U_{31}+EF_1)\,=\,\Pi_3 \tag{28}$$

$$-EKT_1\,e^{-ah}-U_{31}h+U_{32}\,=\,\Upsilon_3 \tag{29}$$

$$-\tilde{\omega}_1T_2-e_{15}U_{31}+\epsilon_{11}F_1=\Delta \tag{30}$$

$$\kappa_{11}^EKae^{ax}(1-K)T_1+\kappa_{11}^EF_1=\Omega \tag{31}$$

By solving the above system of equations in the unknown function of τ,

$$(T_1,\,T_2,\,F_1,\,F_2,\,U_{3\,1},\,U_{3\,2})\,,$$

one finds expressions for the coefficients (which are very long and complex; but we need them in a simple case, see below). By replacing them in the general solution one then obtains the particular solution corresponding to the selected boundary conditions.

As we aim to exhibit the difference of electric potential between the two faces of the piezoelectric panel $\mathcal{P}$ when it is (i) free from stress at $x_1\,=\,h$ where (ii) φ vanishes since the electrode is connected with the ground, (iii) welded to a rigid body at $x_1\,=\,-h$ $(iv-v)$ with zero normal electric displacement and heat conduction, we choose the particular solution corresponding to very simple boundary data:

$$\Phi=0,\quad \Upsilon_3=0,\quad \Pi_3=0,\quad \Omega=0,\quad \Delta=0.$$

For the temperature and electric potential within the panel, which are solution to the considered boundary-value problem, we obtain the expressions

$$T(x_1,\,\tau)=[\frac{\tilde{\omega}_1}{A}(e^{a(x_1-h)}-1)+1]\Theta \tag{32}$$

$$\varphi(x_1,\,\tau)=K\frac{\tilde{\omega}_1}{A}[e^{a(x_1-h)}+a(1-K)x_1-1-ha(1-K)]\Theta \tag{33}$$

In particular (32, (33) respectively give

$$T(h,\,\tau)=\Theta$$

$$T(-h,\,\tau)=[\frac{\tilde{\omega}_1}{A}(e^{-2ah}-1)+1]\Theta \tag{34}$$

$$\varphi(h,\,\tau)=0$$

$$\varphi(-h,\,\tau)=K\frac{\tilde{\omega}_1}{A}[e^{-2ah}-2ah(1-K)-1]\Theta \tag{35}$$

Thus

$$\varphi(-h,\,\tau)-\varphi(h,\,\tau)=B\Theta, \qquad (36)$$

with $B := K\frac{\tilde{\omega}_1}{A}[e^{-2ah}-2ah(1-K)-1]$ a material constant.

Equality (36) shows that the difference of electric potential between the bounding planes of the panel is proportional to the temperature on the upper bounding plane of $\mathcal{P}$, which is due to the sun exposure.

Now we can solve the remaining two equations (14)-(15) joined to the other four boundary data.

2.1.4 B-v problem for a plate perpendicular to the poling direction

Now let us consider a plate filled with the aforementioned material and with the poling direction x_3 perpendicular to the plane of the plate. Referring only to thickness static processes of the form

$$T=T(x_3)\,,\quad \varphi=\varphi(x_3)\,,\quad u_i=u_i(x_3)\,, \qquad (37)$$

the equilibrium field equations (8)-(12) reduce to

$$c_{44}u_{1,\,33}\,=\,0 \qquad (38)$$

$$c_{44}u_{2,\,33}\,=\,0 \qquad (39)$$

$$c_{33}u_{3,\,33}\,+\,e_{33}\varphi_{,\,33}\,=\,0 \qquad (40)$$

$$e_{33}u_{3,\,33}\,+\,\tilde{\omega}_3\,T_{,\,3}\,-\,\epsilon_{33}\varphi_{,\,33}\,=\,0 \qquad (41)$$

$$-\kappa_{33}\,T_{,\,33}\,+\,\kappa^E_{33}\,\varphi_{,\,33}\,=\,0\,. \qquad (42)$$

We note that by the replacements

$$c_{44}\,\to\,c_{11} \qquad (43)$$

$$c_{44}\,\to\,c_{66} \qquad (44)$$

$$(c_{33},\,e_{33})\,\to\,(c_{44},\,e_{15}) \qquad (45)$$

$$(e_{33},\,\tilde{\omega}_3,\,\epsilon_{33})\,\to\,(e_{15},\,\tilde{\omega}_1,\,\epsilon_{11}) \qquad (46)$$

$$(\kappa_{33},\,\kappa^E_{33})\,\to\,(\kappa_{11},\,\kappa^E_{11})\,, \qquad (47)$$

$$\qquad (48)$$

made in the respective lines, these equations become equations (14)-(18); hence all the results established in the previous sections for thickness solutions depending on x_1 can be adapted for the present case.

3 Conclusion

The results explained in the previous sections (36) show that the thermic exposure on the upper plane in certain piezoelectric panels generates a difference of electric potential (36) between the two bounding planes which is proportional to the boundary temperature.

References:

[1] H.F. Tiersten, *Linear Piezoelectric Plate Vibrations,* Plenum–New York 1969