A new concept for the geometry of hierarchically higher biological structures beyond the cell level. The role of Irrational numbers

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Abstract: - This paper carries some new ideas on how to treat the geometry of biological structures involved in Environmental Health Engineering. The term fractal is revisited and subjected to a criticism as far as the irregularity is concerned. Instead of fractal the term φ-geometry is adopted. As a case study the lung system is considered.

Key-Words: - fractal geometry, fractal lung, environmental health engineering, irrational numbers, golden section number, square root of two, square root of three

1 Introduction

In the respiratory system. the air travels down through the windpipe, or trachea, which divides into two branches called bronchial tubes, one entering each lung. The bronchi branch again and again inside the lungs, similar to the branching pattern of a tree, finally becoming hair-like tubes called terminal bronchioles. At the end of the terminal bronchioles are microscopic bubble-like structures called alveoli**.** Each lung contains 300 million alveoli

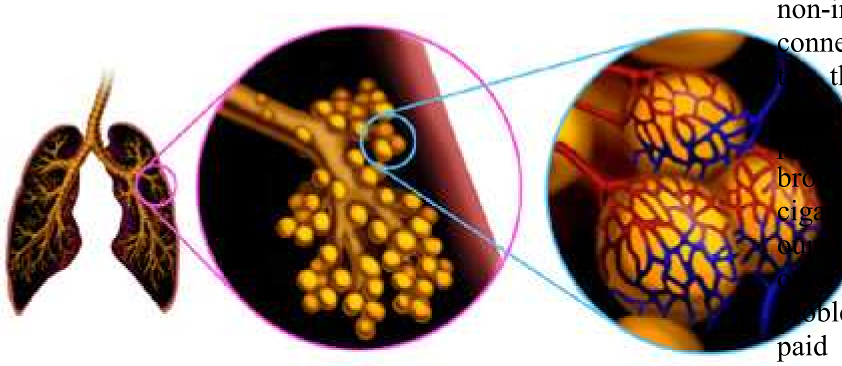

Fig.1 Scaling up the lung geometry

Other things in nature branch out to increase surface area, such as tree roots. This greatly increases the surface area where oxygen can be absorbed; much more than if the lungs were one hollow chamber. Tiny thin-walled blood capillaries surround each alveolus. Oxygen passes through the walls of the alveoli and into the capillaries and is carried throughout the body. Carbon dioxide and water, diffuse from the cells through the capillary walls and into the blood. They are carried back to the alveoli and exhaled from the lungs.

The lung has been modelled in various ways by other researchers . *Anno Domini* 2000 Verbeken [1] compares convential and fractal lung models. He mentions Mandelbrot because of the obvious aspects of self similarity and the non-integer “fractal” dimensions. It connects the type of the modelling ... the disease effect on the ...etry of the lung. He was ...arly interested in smoke bro...olitis apparently caused by ciga...e consumption. Of course in ou...vironment there are actual ... ‘smokes’ too. So the structure ...oblem is very important. Others paid attention to the fact that the trachea is divided in two branchial tubes of unequal size and recently have proposed a double fractality model [2] Already in a brief communication from *anno Domini* 1986 it has been recognised that the irregular branching of the bronchial tree in mammalian species is consisted with a process of morphogenetic self-similarity described by Fibonacci scaling. [3]

2 Problem Formulation

The last sentence of the Introduction brings us in the heart of the problem needed to be solved. If the Fibonacci scaling is followed by the living systems then the golden φ-geometry is the ideal geometry to which the structures in Nature follow, being highly Euclidean and no fractal as Mandelbrot claims. Deviation from the ideal golden geometry could be arise because of the diseases and the general tendency of the things to decay (2^{nd} Law). But among the Irrational numbers followed in the Nature is only the golden section number 1,61803…. Or are there other irrational number to be considered as "fractal dimensions", such as the square roots of 2 and 3? This will be dealt in subsection 3.1. In subsection 3.2 the mathematical properties of the irrational numbers are explored and the simplicity for the mathematical modeling derived from this fct is highlighted. In subsection 3.3 equations of mass transfer in simple phi-geometry structures are presented. Useful equations for the practical engineer are possible to be derived for the overall mass transfer coefficient in series or in parallel or in branched transport problems when dealing with systems such as hierarchically higher beyond the cell level, structures

3 Problem Solution

In Fig.2 the trachea of a healthy lung is branched according to the principle of the golden geometry in extreme and medium ratio. Thus, if d is the trachea diameter and d_1 and d2 are the diameters of the two bronchial pipes then the following equation (1) is valid

$$d_1/d = d_2/d_1 \qquad (1)$$

Because also d is the sum of d_1 and d_2 then the golden section number appears in the ratios of the equation 1. At least we could consider this fact in order to approximate the lung geometry. Indeed other researchers with the fractal approach they have announced fractal dimensions for the juvenile rats of the order of the phi number.
1.602 for young females and 1.626 for young males. These two values within the experimental error are good approximations of the 1.618…number. In the case of adults there is a slight deviation (1.588 and 1.547 respectively). But this, it could be expected as a consequence of the age. Besides the phi there are other two irrational numbers that could be derived from Fibonacci- like sequences and appear in self similar yet regular structures.

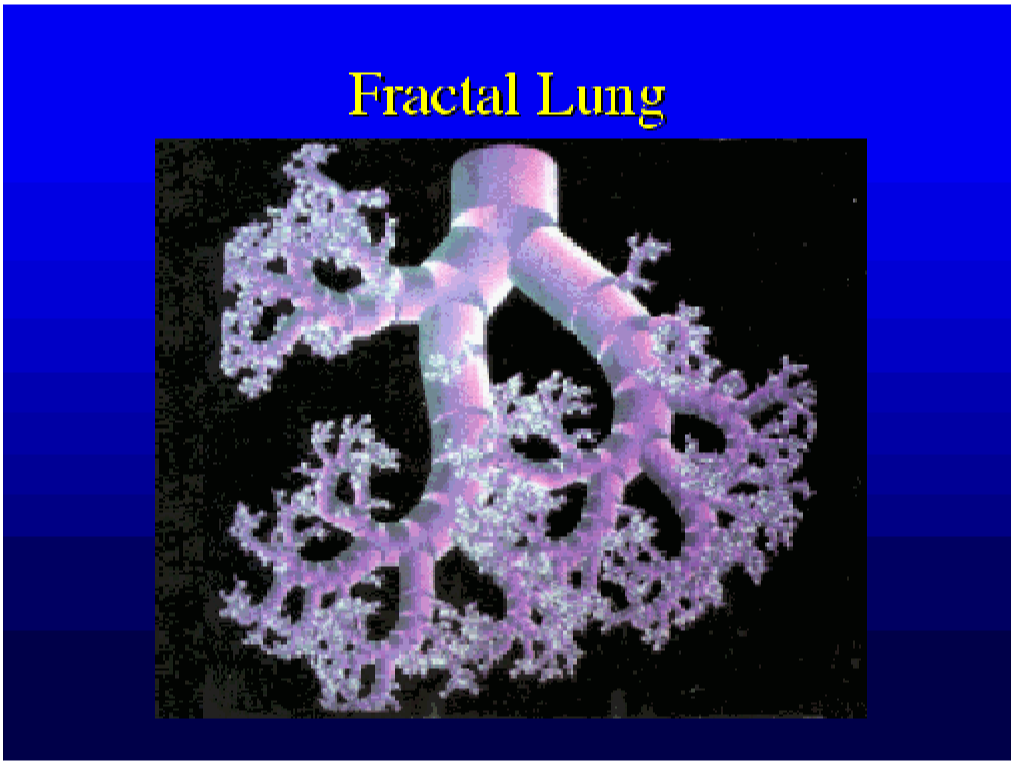

Fig. 2 The asymmetry of the bronchial tree branching is clearly showed in this picture.

3.1 Deriving the Irrational numbers from Integer Sequences

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3.1.1 The case of square root of 2

.It is well known the property of the Fibonacci integer sequence
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377… (IS-1)

$a_1=a_2=1$
$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n>2 \qquad (2)$$

to give good approximations of the golden-section number φ , when the ratios of subsequent terms are considered. Thus a simple sequence of integers produces irrational or asymmetric numbers. As Plato put it "everywhere in the Universe the Rational is coupled with the Irrational". In the Academy it was most probably known an integer sequence which, in the same manner as the Fibonnacci sequence, could produce as a limit another famous irrational number, the $\sqrt{2}$. We have tried to reconstitute the integer sequence with the property the subsequent terms to give good approximations to the square root of 2 alternately from higher and lower values respectively.
Indeed the following integer sequence:

1, 1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, 169, 239… (IS-2)

resembling the Fibonnacci sequence, but created slightly differently with
$a_1=a_2=1$
and for n>2:
$a_n = a_{n-1} + a_{n-2}$ for n=2k+1 and
$$a_n = a_{n-1} + a_{n-3} \quad \text{for } n=2k \qquad (3)$$

gives the following property

$$\frac{1}{1} < \frac{7}{5} < \frac{41}{29} <$$
$$\frac{239}{169} < .. < \sqrt{2} < .. \frac{99}{70} < \frac{17}{12} < \frac{3}{2}$$

Notice that there is a value of n for which each of the integer sequences IS-1 and IS-2 the n_{th} term equals the square of n:

$$a_n = n^2. \qquad (4)$$

In the Fibonnacci sequence this number is n =12 and for the "retarded Fibonnacci" sequence (IS-2) n is equal to 13.

We have created more integer sequences which present this property. Also these integer sequences have the property to "tend" to the value of an irrational number. The integer sequences are shown in Table 1.

Table 1 1
Integer sequences which for a certain value of n, a_n equals the n squared

The first 14 terms of each IS

2, 7, 9, 16, 25, 41, 66, 107, 173, 280, 453, 733, 1186, 1919 IS-3

2, 6, 8, 14, 22, 36, 58, 94, 152, 246, 398, 644, 1042, 1686 IS-4

5, 3, 8, 11, 19, 30, 49, 79, 128, 207, 335, 542, 877, 1414 IS-5

3, 4, 7, 10, 17, 27, 37, 64, 101, 138, 239, IS-6

1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, IS-7

2, 1, 3, 4, 8, 12, 20, 40, 60, 100, 200, 300, 500, 1000 IS-8

1, 1, 2, 3, 5, 12, 17, 29, 46, 75, 121, 288, 409, 697 IS-9

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 477 IS-1

1, 1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, 169, 239 IS-2

The three first IS's of the Table 1 follow a Fibonnacci-like formula of creation

$a_n = a_{n-1} + a_{n-2}$ for $n>2$. (5)

The difference is that a_1 and a_2 are different. Instead of the Fibonnacci-corresponding values 1 and 1, the starting values of the three sequences are as follows:
in the IS-3 sequence $a_1 = 2$ and $a_2 = 7$.
In the IS-4 sequence $a_1 = 2$ and $a_2 = 6$.
In the IS-5 sequence $a_1 = 5$ and $a_2 = 3$.
In all those three sequences the subsequent-term ratios tend to φ , as in the Fibonnacci sequence.

The IS-6 sequence follows the rule:.

For n>2
$a_n = a_{n-1} + a_{n-2}$ for n=3k+2 and n=3k
$a_n = a_{n-1} + a_{n-3}$ for n=3k+1 (6)

In an analogous way the other sequences IS-7 through IS-9 could be produced.

A3.1.2 The case of square root of 3

Archimedes left an approximation for $\sqrt{3}$

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

This puzzles the modern reader, how did he get this approximation from?
Possibly he did get this from the integer sequences below

1, 1, 2, 3, 5, 8, 11, 19, 30, 41, 71, 112, 153, 265, 418, 571, 989…. (IS-10)

1, 2, 3, 4, 7, 11, 15, 26, 41, 56, 97, 153, 209, 362, 571, 780, 1351 (IS-11)

The two sequences differ only to the second term. The generating procedure for both is

$a_n = a_{n-1} + a_{n-2}$ for n=3k+2 and n=3k
$a_n = a_{n-1} + a_{n-3}$ for n=3k+1 (7)

same as for the sequence IS-6 mentioned above. Easily we can see that the ratios of the terms

$\frac{a_{3k+2}}{a_{3k+1}}$ tend to $\sqrt{3}$ i.e. they are approximations of $\sqrt{3}$. The same happens with the sequence IS-6.

A further generalization might exist for the square roots of higher integers (.>3). We are working also towards that direction, besides exploring further the connection between biological patterns and the square roots of 2 and 3.

Integer sequences following the same generating rule lead to the same irrational number, regardless the first two terms. The integer sequences generated so far could be classified into three groups.

1. those following the $a_n = a_{n-1} + a_{n-2}$. They are associated with φ, the same as the Fibonnacci sequence

2. those following $a_n = a_{n-1} + a_{n-3}$ for n=2k and $a_n = a_{n-1} + a_{n-2}$ for n=2k+1 associated with the square root of 2, and finally

3 those following $a_n = a_{n-1} + a_{n-2}$ for n=3k+2 and n=3k and $a_n = a_{n-1} + a_{n-3}$ for n=3k+1 associated with the square root of 3

3.2 Mathematical properties of the Irrational numbers

For φ
There are well known the remarkable properties of the golden section number φ. Starting from:

$$1+\varphi = \varphi^2 \quad (8)$$

$$\varphi + \varphi^2 = \varphi^3 \quad (9)$$

any power of this number or products of powers could simplify to linear forms. Also there is known another property for the reciprocals, as below:

$$\frac{1}{\phi}+\frac{1}{\varphi^2}+\frac{1}{\varphi^3}+\ldots=\varphi \quad (10)$$

From the golden triangle and the associate golden rectangle the golden helix can be derived followed by certain living creatures, figures 3 and 4 below

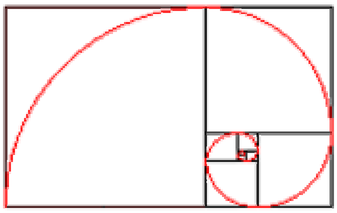

Fig.3 The golden helix

The trigonometric properties arising from the golden triangle (the isosceles triangle with angles 36°, 72° and 72°) in Fig.4, they are very interesting because they are functions of φ.

$$\cos(36^o) = \sin(54^o) = \varphi/2 \quad (11)$$

$$\cos(72^o) = \sin(18^o) = (1-\varphi)/2 \quad (12)$$

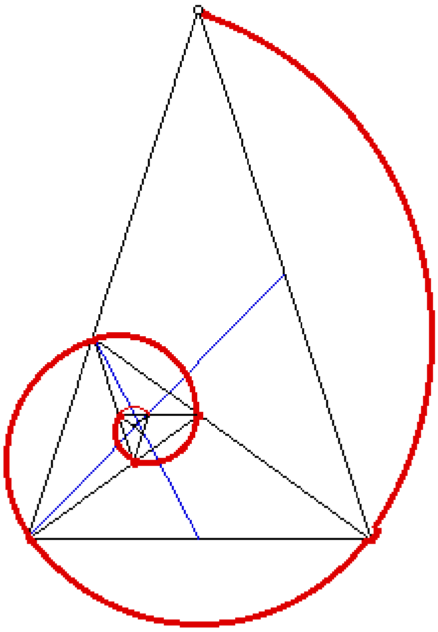

Fig. 4 The golden phi triangle and the helix created.

Right helices have been in this work constructed, on the basis of the $\sqrt{2}$ and $\sqrt{3}$.

For the $\sqrt{2}$:
The generation rule is: Start with a right isosceles triangle of side 1, the hypotenuse being $\sqrt{2}$. Then the hypotenuse becomes the vertical side of the new triangle, so that its hypotenuse become 2. 2 becomes the side of the next isosceles right triangle and so on. In this series of triangles the hypotenuse value alternates between an irrational and a rational number. The ratio of two subsequent-triangle hypotenuses is $\sqrt{2}$.

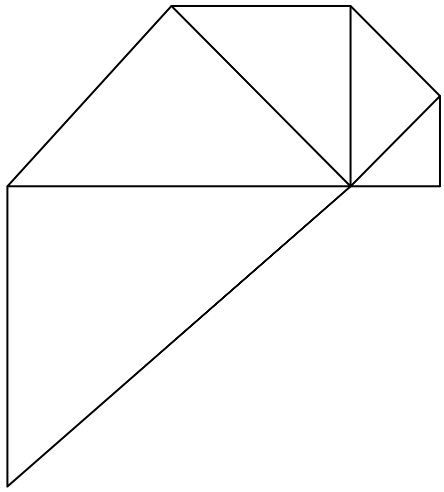

Fig.5. The helix of square root of 2

For the $\sqrt{3}$:

The generation rule is: Start with a right triangle of sides 1 and $\sqrt{3}$ The hypotenuse is of course 2. Then the next (second) triangle has sides 2 and $2\sqrt{3}$.The next (third one) has sides 4 and $4\sqrt{3}$ and so on.
It is shown that the hypotenuses of the series of the triangles follow the simple integer sequence:

$$2, 2^2, 2^3, \ldots\ldots 2^n, \ldots\ldots$$

Another property is that in the picture of the helix$\sqrt{3}$ is that certain diagonals give multiples of 3n leaving as rest factors prime numbers etc.

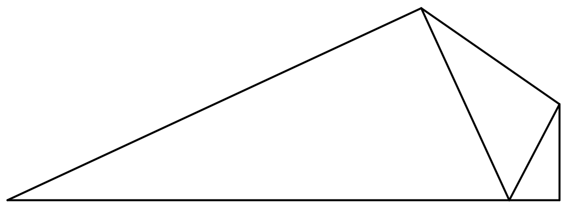

Fig.6 The right helix of the square root of 3

With the irrational numbers there is a rich kingdom of properties revealed or to be revealed, connecting ancient wisdom [5] with modern mathematical capacity The following also is valid: if the pattern followed is the phi one, the angles involved are 36, 54, 72 degrees. The square root of two pattern is associated with angles of 45 degrees and the square root of three with 30 and 60 degrees, all of well known sinus and co-sinus.

3.3 Deriving Transport equations based on the simplicity of the phi-geometry

Useful for the practical Engineer equation of transport include the overall mass- or heat- transfer coefficients K or U in a simplified geometry (of already examined plate, cylinder , sphere) and in a prevailing dimension [6] . Also phases in series and or in parallel have been already examined.[7]
For biological structures the tendency is to show well controlled therefore constant and relatively small fluxes in order to avoid fouling phenomena on the cell membranes. This can be obtained with self similarity accompanied by regularity and NOT irregularity as the adepts of Mandelbrot accept.

In a φ structure or in a branched structure in general a 2-D model could consider a combination of the features of the in series and the in parallel cases, with the trigonometric functions of the angles with the well known properties are involved.
With the phi structure and branches in parallel

$$K_{overall} = K + \frac{D_{eff}}{d_1} + \frac{D_{eff}}{d_2} + \ldots \frac{D_{eff}}{d_n} + \ldots \qquad (13)$$

where K overall the overall mass transfer coefficient with K the local boundary layer mass transfer coefficient , D_{eff} the effective diffusion coefficient implying that also other mechanisms besides molecular diffusion could be involved and d_1 d_n the diameters of the branches.

In this case he sum of the reciprocals of φ appears in the equation 13 easily to be thus simplified.

In the in-series connection mode the equation for the overall mass transfer coefficient it is according to what is known from the Transport Phenomena Theory:

$$\frac{1}{K_{overall}} = \frac{1}{K} + d_1\left(\frac{1}{K} + d_2\left(\frac{1}{K} + \ldots d_{n-1}\left(\frac{1}{K} + \frac{d_n}{D_{eff}}\right)\ldots\right)\right) \qquad (14)$$

In this in series- case if the structure is a golden section one , it appears in the equation (14) sum of products of φ , therefore the simplification is again possible.

In our laboratory we are currently elaborating the model with 2 prevailing dimensions (2-D geometry) for a branched φ structure, where the equations (13) and (14) could be combined and furthermore the trigonometric functions of the golden angles mentioned above could be introduced.

4 Discussion -Conclusion

This paper is a product of seeking the true geometry of structures that appear in Environmental Health Engineering focusing to the lung. On working this task a more general challenge had to be faced. Namely, which are the true geometry characteristics of living entities, and not only? The axiomatic answer given is that the structures follow irrational numbers , in the vast majority, the golden section number φ with its remarkable algebraic and trigonometric properties. Indeed there is an enormous evidence for this to happen and not only with the living matter. The planets orbit the Sun following the Pythagorean ancient knowledge , recently decrypted by Ippokrates Dakoglou [8] Plants and animals in numerous examples present rations of the golden geometry. Also the DNA molecule does the same. Now the following conclusion could be drawn, If in the Macro-Cosmos and as well in the Micro-Cosmos the golden section prevails , this is expected to happen also to the intermediate scales, where the organs of the human and other animals body belongs.

Irrational numbers , besides φ also the square roots of 2 and 3, arise from simple integer sequences. For φ it was already known the integer sequence "producing" it. For the square roots of 2 and 3 , it is the merit of the present work to identify the Integer Sequences associated. Thus the rations given by Plato and Archimedes find their place.

Furthermore the three irrational numbers here considered are associated with:

$\sqrt{2}$: the rectangle

φ: the normal pentagon

$\sqrt{3}$: the normal hexagon.

So that all the main symmetries of Nature are covered by considering those irrational numbers.[9]

We have coined the term golden geometry or φ-geometry, opposed to Mandelbrot's fractal geometry thus defending Euclidean Geometry , which is followed by Nature, i.e. highlighting the fact that biological structures are self similar (as Mandelbrot says) but NOT irregular: They follow distinct patterns. Due now to those remarkable properties of the irrational numbers, the number φ in particular, the problem faced in the Environmental Health Engineering Modeling can also be simplified, the yield being simple transport equations helpful for the practical Environmental Engineer and other scientists.

This is the outcome of the research carried out in our Laboratory of Transport Phenomena, so far. In the continuation the aim is twofold , at concretizing these new ideas to the specific problems of pollutant transport in higher forms of living tissues such as the lung and at expanding the findings in cases where other irrationals, except the golden section number phi, rule.

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