Analytical Description of Transient Currents in Transmission Lines with Phase Delay and Inductances

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Abstract: - In this paper, a method is proposed in order to analyze transient currents in three-phase transmission lines subject to phase shift and inducting loads. The procedure uses a mathematical dynamical model to describe the system, consisting in an homogeneous multidimensional differential equation containing an inner (i.e., in the state) point delay.

Key-Words: - Time-Delay System; Phase-Shift; Delay Differential Equation; Nonlinear Transmission Line.

1 Problem Statement

The analysis of electric signals in transmission lines loaded by nonlinear elements has received increasing interest during the last decade. In many applications, the signal distortion provoked by lossy lines and skin effect constitutes an important problem; thus, many authors have addressed analytical descriptions for the dynamic behaviour of lossy transmission lines with nonlinear elements. Those descriptions can be classified in two main cathegories: the so-called «all-numerical» approaches [1] (leading sometimes to a poor insight of the transient process), and mixed approaches (which involve a frequency domain description for the linear elements of the system, as well as characteristic equations in the time domain for the nonlinear elements). For an excellent critical review of the most outstanding mixed methods available, see [2]. An improved mixed method has been recently developed in [3]. Transient simulation using mixed methods has been provided in [11].

Within the above mentioned first cathegory, many authors have dealt with the method of characteristics [4] [5] [6]. In [7], the reader will find a very useful and clarifying work on the uniqueness properties for solutions of transmission lines in the presence of nonlinear resistors. Some further properties of the dynamic behaviour in the presence of nonlinear resistors can be found in [8] and [9]. In these mentioned works, however, the transmission line is supposed to be linear, and nonlinearities are assumed to be found in the loads, exclusively.

Nonlinear distortion of the transmitted signals in a line is a reported physical phenomenon [10]. The introduction of phase delay in an electric current can be modelled by using an amplifier and a phase shift with line impedances consisting in linear resistors, as in the equivalent electric circuit for the Minorsky problem [13], and a linear inductance as a general load for the circuit.

The differential equation governing the electric current in that circuit in the homogeneous case, according to [15], is given by

$$L\frac{di(t)}{dt}+(R+R_0)i(t)+\lambda R_0 i(t-h)=0 \quad (1)$$

which is a delayed differential equation (DDE) with delay h>0. This kind of equations is quite common when modelling precisely propagation phenomena of electric nature, including transmission lines. See, for instance, [16] [17].

This note uses a delayed time-domain model of the transmission line in order to take into account the phase distortion in the transmitted current (Section II). After that, a simple mathematical method is developed in order to describe exactly the transients for coupled three-phase transmission lines when the load is supposed to be a pure inductance (Section III). Section IV shows some simulation results.

2 Model for the transmission line

According to the model suggested in the previous section, a time-delayed three phase circuit is proposed in order to model a three phase transmission line with phase distortion and inducting loads.

Due to the inducting nature of the load, it is convenient to take into account mutual inductance phenomena produced in the other phases by the electric current along a phase. It is assumed that the first derivative of i1 is augmented by a term

$$\alpha_1(i_2(t)+i_3(t)) \quad (2)$$

with α_1 relatively small, and $[\alpha_1]$ = R L^{-1}. Similarly for the other currents. Define the following quantities:

$$Z_1=-\frac{1}{L}(R_1+R_0);\quad Z_2=-\frac{1}{L}(R_2+R_0);\quad Z_3=-\frac{1}{L}(R_3+R_0);\quad Z_\lambda=-\frac{\lambda R_0}{L} \quad (3)$$

Taking into account equations (2) and (3), the following delay differential equation is obtained as a dynamic model in the time domain for the three phase line:

$$\begin{bmatrix}\frac{d}{dt}i_1(t)\\ \frac{d}{dt}i_2(t)\\ \frac{d}{dt}i_3(t)\end{bmatrix}=\begin{bmatrix}Z_1 & \alpha_1 & \alpha_1\\ \alpha_2 & Z_2 & \alpha_2\\ \alpha_3 & \alpha_3 & Z_3\end{bmatrix}\begin{bmatrix}i_1(t)\\ i_2(t)\\ i_3(t)\end{bmatrix}+\begin{bmatrix}Z_\lambda & 0 & 0\\ 0 & Z_\lambda & 0\\ 0 & 0 & Z_\lambda\end{bmatrix}\begin{bmatrix}i_1(t-h)\\ i_2(t-h)\\ i_3(t-h)\end{bmatrix} \quad (4)$$

Equation (4) describes an invariant linear system with a point delay in its state. For the sake of simplicity, define

$$D_0=\begin{bmatrix}Z_1 & \alpha_1 & \alpha_1\\ \alpha_2 & Z_2 & \alpha_2\\ \alpha_3 & \alpha_3 & Z_3\end{bmatrix};\quad D_1=\begin{bmatrix}Z_\lambda & 0 & 0\\ 0 & Z_\lambda & 0\\ 0 & 0 & Z_\lambda\end{bmatrix};$$

$$I(\cdot)=\begin{bmatrix}i_1(\cdot)\\ i_2(\cdot)\\ i_3(\cdot)\end{bmatrix} \quad (5)$$

Therefore, for null interval initial conditions (i.e., I(t) = 0, t < 0) and punctual initial conditions

$$I(0)=I_0=\begin{bmatrix}i_1(0)\\ i_2(0)\\ i_3(0)\end{bmatrix} \quad (6)$$

then one obtains the following homogeneous and invariant linear system with a point delay in its state:

$$\dot{I}(t)=D_0I(t)+D_1I(t-h) \quad (7)$$

3 Exact solution for homogeneous case

Homogeneous initial conditions for the currents along the line have been assumed. This approach is not unusual in the field (see [9], Section III).

In many works, several formulae for solving delayed integro-differential systems have been provided. The most general of such results can be seen in [12]. However none of the exact proposed solutions of general nature is explicit. This fact implies that the exact solutions always include a subsidiary delayed differential equation that is not directly solvable to compute the fundamental matrix [12] [13].

For t such that 0 ≤ t ≤ h system (7) is reduced to the non-delayed system $\dot{I}(t)=D_0I(t)$ with initial condition (6) which can be solved as an ordinary differential equation. Suppose that $\phi_1(t)$ is the solution for such a case. In addition, note that for t such that h < t ≤ 2h Eq. (7) becomes

$$\dot{I}(t)=D_0I(t)+D_1\phi_1(t-h) \quad (8)$$

Again it could be possible to solve (8) as an ordinary differential equation with a known forcing term, and find an exact explicit solution $\phi_2(t)$ of I(t) for t such that h < t ≤ 2h. By following succesively this simple method it would be possible to find the exact explicit solution for any interval. The problem then is to find a general formula for the n-th time interval of t, (n-1)h < t ≤ nh . This is an extension of the so-called method of steps [14].

In order to describe system (7) in an analytic way, an exact solution will be computed for each time interval according to the following result.

Theorem (main result). Consider system (7). An exact explicit unique solution for that system on [0, ∞), subject to null interval initial conditions (i.e., I(t) = 0, t < 0) and punctual initial conditions (6), provided that the matrices of the pair $(D_1, e^{-D_0 h})$ commute, is built in closed form by using truncated functions as follows

$$\Omega(t) = \sum_{i=1}^{\infty} \phi_{iT}(t), \quad all\ t \geq 0 \quad (9a)$$

where $\phi_{1T}(t) = \left(U(t) - U(t-h)\right)e^{D_0 t} I_0$ (9b)

$$\phi_{iT}(t) = \left(U(t-(i-1)h) - U(t-ih)\right)e^{D_0 t} \cdot \left[E + \sum_{k=1}^{i-1}\left[(t-kh)^k \prod_{j=1}^{k}\left(\frac{D_1 e^{-D_0 h}}{j}\right)\right]\right] I_0 \quad (9c)$$

being U(t) the unity step function at t = 0, and E the identity matrix.

For the proof, see the Appendix.

Remark: Note that this result is relevant to system (7). Matrix D1 is diagonal, and therefore it commutes with $e^{-D_0 h}$, provided that Z_λ has the same value for each one of the three phases in the transmission line, which is an assumption explicitly made in Eq. (4), being this one the condition for the applicability of the theorem. The commutativity holds, in particular, if the load is symmetric with identical passive elements.

4 Simulation results

Simulation results by using expression (9) have been obtained by using Mathematica software with the following standard parameter values: L = 1H, R_1 R1 = 2K, R_0 = 20K, λ = 0.01, I(0) = (20, -20, 0), h = 0.001, $\alpha_1 = \alpha_2 = \alpha_3$ = -1000, which implies $Z_1 + Z_2 + Z_3$= -22000, Z_λ= -200. Figures 1, 2 and 3 show the time evolution of the three phase currents in the transmission line for t > 0.2ms. Observe the peaks registered after the passing of 1 milisecond and 2 miliseconds, corresponding to the inner time delay of the system. Although the third phase has null initial conditions, a very small effect can be observed in its evolution due to interconnection with the other two phases with non zero initial conditions.

5 Conclusions

This note has introduced an analytical method to describe explicitly a kind of delay-differential systems, i. e., three-phase transmission lines subject to phase shift and inducting loads. The analytical expression obtained to describe the transient response of the system is exact, as stated in the main result, which provides a suitable tool for dealing with this kind of systems. The method can be extended to cover other configurations by appropriately rewriting the conditions of Theorem 1.

Appendix: Proof of Theorem 1

Proof (By complete induction)

Let us consider first n = 1 . Therefore t is such that 0 ≤ t < h . Eq. 9 for this case becomes

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$$\Omega(t) = \phi_{1T}(t) = e^{D_0 t} I_0 = \phi_1(t) \quad (A1a)$$

and Eq. 7 becomes

$$\dot{I}(t) = D_0 I(t), \qquad I(0) = I_0 \quad (A1b)$$

Substitution of (A1a) in (A1b) proves that Ω is a solution for n = 1.

Assume now n = 2. Therefore t is such that h ≤ t < 2h. Eq. (9) for this time interval becomes

$$\begin{aligned}\Omega(t) &= e^{D_0 t}\left[E + (t-h)D_1 e^{-D_0 h}\right] I_0 \\ &= e^{D_0 t} I_0 + (t-h)e^{D_0 t} D_1 e^{-D_0 h} I_0\end{aligned} \quad (A2a)$$

and Eq. (7) becomes

$$\dot{I}(t) = D_0 I(t) + D_1 \phi_1(t-h) = D_0 I(t) + D_1 e^{(t-h)D_0} I_0$$
(A2b)

Substitution of (A2a) in the left-hand side of (A2b) yields

$$\begin{aligned}\dot{I}(t) &= D_0 e^{D_0 t} I_0 + e^{D_0 t} D_1 e^{-D_0 h} I_0 \\ &+ (t-h) D_0 e^{D_0 t} D_1 e^{-D_0 h} I_0\end{aligned} \quad (A3a)$$

In turn, substitution of (A2a) in the right-hand side of (A2b) yields

$$\dot{I}(t) = D_0 e^{D_0 t} I_0 + (t-h) D_0 e^{D_0 t} D_1 e^{-D_0 h} I_0 + D_1 e^{D_0 t} e^{-D_0 h} I_0 \quad \text{(A3b)}$$

Then (A2a) is a solution for (A2b) - i.e., (A3a) = (A3b) - if the matrices of the pair $(D_1, e^{-D_0 h})$ commute, which is the case because the first element of the pair is a diagonal matrix.

Now assume that the proposed solution (9) is valid for both n-2 and n-1, this is, for t such that (n-3)h ≤ t < (n-2)h and for (n-2)h ≤ t < (n-1)h . Thus

$$\phi_{n-2}(t) = e^{D_0 t}\left[E + \sum_{i=1}^{n-3}\left[(t-ih)^i \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \quad \text{(A4)}$$

is a solution for system (7) for any t such that (n-3)h ≤ t < (n-2)h , and

$$\phi_{n-1}(t) = e^{D_0 t}\left[E + \sum_{i=1}^{n-2}\left[(t-ih)^i \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \quad \text{(A5)}$$

is a solution for t such that (n-2)h ≤ t < (n-1)h . In particular, $\phi_{n-1}(t)$ satisfies the differential equation (7) for such interval:

$$\dot{\phi}_{n-1}(t) = D_0 \phi_{n-1}(t) + D_1 \phi_{n-2}(t-h) \quad \text{(A6)}$$

Note that, by differentiating $\phi_{n-1}(t)$ in (A5) one obtains

$$\dot{\phi}_{n-1}(t) = D_0 \phi_{n-1}(t) + e^{D_0 t}\left[\sum_{i=1}^{n-2}\left[i(t-ih)^{i-1} \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \quad \text{(A7)}$$

Substitute (A7) in (A6) to yield

$$D_1 \phi_{n-2}(t-h) = e^{D_0 t}\left[\sum_{i=1}^{n-2}\left[i(t-ih)^{i-1} \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0$$

$$\Rightarrow D_1 e^{D_0(t-h)} I_0 + D_1 e^{D_0(t-h)}\left[\sum_{i=1}^{n-3}\left[(t-(i+1)h)^i \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 = e^{D_0 t}\left[\sum_{i=1}^{n-2}\left[i(t-ih)^{i-1} \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0$$

(A8)

The proposed solution (9) for the n-th interval (n-1)h ≤ t < nh becomes

$$\Omega(t) = \phi_n(t) = e^{D_0 t}\left[E + \sum_{i=1}^{n-1}\left[(t-ih)^i \prod_{j=1}^{i}\left(\frac{D_1 e^{-D_0 h}}{j}\right)\right]\right] I_0$$

(A9)

and therefore it must be demonstrated that (A9) satisfies the differential equation as follows

$$\dot{\phi}_n(t) = D_0 \phi_n(t) + D_1 \phi_{n-1}(t-h) \quad \text{(A10)}$$

Rewrite the right-hand side of (A10) as

$$D_0 \phi_n(t) + D_1 \phi_{n-1}(t-h) = D_0 \phi_n(t) + D_1 e^{D_0(t-h)} I_0 + D_1 e^{D_0(t-h)}\left[\sum_{i=1}^{n-2}\left[(t-(i+1)h)^i \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \quad \text{(A11)}$$

By substracting and adding the left-hand and the right-hand sides of identity (A8) to (A11), taking into account the commutativity of the matrices of the pair $(D_1, e^{-D_0 h})$, and taking derivatives in (A9), then one obtains the next chain of identities

$$\begin{aligned} D_0 \phi_n(t) + D_1 \phi_{n-1}(t-h) &= D_0 \phi_n(t) + D_1 e^{D_0(t-h)} I_0 + D_1 e^{D_0(t-h)}\left[\sum_{i=1}^{n-2}\left[(t-(i+1)h)^i \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \\ &\quad - D_1 e^{D_0(t-h)} I_0 - D_1 e^{D_0(t-h)}\left[\sum_{i=1}^{n-3}\left[(t-(i+1)h)^i \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \\ &\quad + e^{D_0 t}\left[\sum_{i=1}^{n-2}\left[i(t-ih)^{i-1} \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \\ &= D_0 \phi_n(t) + D_1 e^{D_0(t-h)} (t-(n-1)h)^{n-2} \prod_{j=1}^{n-2} \frac{1}{j} D_1 e^{-D_0 h} I_0 \\ &\quad + e^{D_0 t}\left[\sum_{i=1}^{n-2}\left[i(t-ih)^{i-1} \prod_{j=1}^{i} \frac{1}{j} D_1 e^{-D_0 h}\right]\right] I_0 \end{aligned}$$

and hence Eq. (9) satisfies (7) for any positive integer n.

Note that a sufficient (but not necessary) condition for satisfying that the matrices of the pair $(D_1, e^{-D_0 h})$ commute is that D1 and D0 commute, which can be checked by expanding exp(-D0h) in powers of D0.

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