# Analysis and Investigation of Reliability Model in Computer Networks

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*Abstract:* We use a mathematical model of an open queueing network in heavy traffic. The law of the iterated logarithm for the extreme virtual waiting time of a customer in heavy traffic in open queueing networks has been presented. Finally, we present an application of the theorem - a reliability model from computer network practice.

*Key–Words:* mathematical models of technical systems, performance evaluation, reliability theory, queueing theory, open queueing network, heavy traffic, the law of the iterated logarithm, virtual waiting time of a customer, extreme value.

## 1 Introduction

One can apply the theory of queueing networks to obtain probability characteristics of technical systems (for example, the reliability function of computer networks).

At first we try to present a survey of papers designated to applying the results of the queueing theory in reliability. In one of the first papers of this kind ([2]), it is investigated the reliability of a distributed program in a distributed computing system and showed that there is a probability that a program which runs on multiple processing elements that have to communicate with other processing elements for remote data files will be executed successfully. In [9], the authors consider a single machine, subject to breakdown, that produces items to inventory. The main tool employed is a fluid queue model with repair. To analyze the performance of multimedia service systems, which have unreliable resources, and to estimate the capacity requirement of the systems, developed a capacity planning model using an open queueing network (see [7]). The paper of [1] discusses a novel model for a reliable system composed of $N$ unreliable systems, which can hinder or enhance one another's reliability. In [11], analyze the behaviour of a heterogeneous finite-source system with a single server. As applications of this model, some problems in the field of telecommunications and reliability theory are treated. In [8], it is investigated the management policy of an $M/G/1$ queue with a single removable and non-reliable server. They use the analytic results of this queueing model and apply an efficient Matlab program to calculate the optimal threshold of management policy and some system characteristics. In the papers of [3, 4], using the law of the iterated logarithm for the queue length of customers, the reliability function of computer network is estimated and a theorem similar as Theorem 2 is proved.

In this paper, we presented the law of the iterated logarithm for the extreme virtual waiting time of a customer in heavy traffic in open queueing networks.

First we consider open queueing networks with the "first come, first served" service discipline at each station and general distributions of interarrival and service time. The basic components of the queueing network are arrival processes, service processes, and routing processes. Particularly, there are $k$ nodes with the node $i$ having a single server and a waiting room of unlimited capacity. The external input stream to the node $i$ is a renewal process, the interarrival time of this process with the mean $\lambda_i = \left(M\left[z_n^{(i)}\right]\right)^{-1} > 0$, and finite variance $a_i = D\left(z_n^{(i)}\right) > 0$, $i = 1, 2, ..., k$. These external input streams at the various nodes are assumed to be independent. The service times at the node $i$ are independent and have a common distribution with the mean $\mu_i = \left(M\left[S_n^{(i)}\right]\right)^{-1} > 0$ and finite variance $\sigma_i = D\left(S_n^{(i)}\right) > 0$, $i = 1, 2, ..., k$. The service times at the node $i$ are also independent of all customer arrivals at the node $i$. A customer leaving the node $i$ is immediately and independently routed to the node $j$ with probability $p_{ij}$; and the customer departs the system from the node $i$ with probability $q_i = 1 - \sum_{j=1}^{N} p_{ij}$. The matrix $P^* \equiv [p_{ij}]$ is called a switching matrix. Observe that this system is quite general, encompassing the tandem system, acyclic networks of $GI/G/1$ queues, and networks of $GI/G/1$ queues with feedback.

Let us define $V_j(t)$ as the virtual waiting time of customers at the $j$-th station of a queueing network at time $t$ (one must wait until a customer arrives at the $j$-th station of the queueing network to be served at time $t$), $\beta_j = \frac{\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij}}{\mu_j} - 1 > 0$, $\hat{\sigma}_j^2 = \sum_{i=1}^{k} p_{ij}^2 \cdot \mu_i \left(\sigma_j + \left(\frac{\mu_i}{\mu_j}\right)^2 \cdot \sigma_i\right) + \lambda_j \cdot \left(\sigma_j + \left(\frac{\lambda_j}{\mu_j}\right)^2 \cdot a_j\right) > 0$, $j = 1, 2, \ldots, k$.

We suppose that the following condition is fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} > \mu_j, \; j = 1, 2, \ldots, k. \qquad (1)$$

Note that this condition quarantees that, with probability one there exists a virtual waiting time of a customer and this virtual waiting time of a customer is constantly growing.

One of the results of the paper is a following theorem on the law of the iterated logarithm for the extreme virtual waiting time of a customer in an open queueing network.

**Theorem 1** *If conditions (1) are fulfilled, then*

$$P\left(\overline{\lim_{t\to\infty}} \frac{\sup_{0\le s\le t} V_j(t) - \beta_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1\right)$$
$$= P\left(\underline{\lim_{t\to\infty}} \frac{\sup_{0\le s\le t} V_j(t) - \beta_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = -1\right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(t) = \sqrt{2t \ln \ln t}$.

**Proof:** This theorem is presented in [5], and we omit the proof.

The proof of theorem is completed.

## 2 Reliability Function of Computer Networks

Now we present a technical example from the computer network practice. Assume that queues arrive at a computer $v_j$ at the rate $\lambda_j$ per hour during business hours, $j = 1, 2, \ldots, k$. These queues are served at the rate $\mu_j$ per hour in the computer $v_j$, $j = 1, 2, \ldots, k$. After service in the computer $v_j$, with probability $p_j$ (usually $p_j \ge 0.9$), they leave the network and with probability $p_{ji}$, $i \ne j$, $1 \le i \le k$ (usually $0 < p_{ji} \le 0.1$) arrive at the computer $v_i$, $i = 1, 2, \ldots, k$. Also, we assume the computer $v_j$ fails when the extreme virtual waiting time of queues is more than $k_j$, $j = 1, 2, \ldots, k$.

In this section we will prove the following theorem on the reliability function of computer network.

**Theorem 2** *If* $t \ge \max_{1\le j\le k} \frac{k_j}{\beta_j}$ *and conditions (1) are fulfilled, the computer network becomes unreliable (all computers fail).*

**Proof:** At first, using Theorem 1 we get for $0 < \varepsilon < 1$ that

$$P\left(\lim_{t\to\infty} \frac{\sup_{0\le s\le t} V_j(t)) - \beta_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon\right) = 0, \quad (2)$$

$j = 1, 2, \ldots, k$.

Let us investigate a computer network which consists of the elements (computers) $v_j$ that are indicators of stations $X_j$, $j = 1, 2, \ldots, k$.

Denote

$$X_j = \begin{cases} 1, & \text{if the element } v_j \text{ is reliable} \\ 0, & \text{if the element } v_j \text{ is not reliable,} \end{cases}$$

$j = 1, 2, \ldots, k$.

Note that $\{X_j = 1\} = \{\sup_{0\le s\le t} V_j(t) < k_j\}$, $j = 1, 2, \ldots, k$.

Denote the structural function of the system of elements connected by scheme 1 from $k$ (see, for example, [6]) as follows:

$$\phi(X_1, X_2, \ldots, X_k) = \begin{cases} 1, & \sum_{i=1}^{k} X_i \ge 1 \\ 0, & \sum_{i=1}^{k} X_i < 1. \end{cases}$$

Denote $y = \sum_{i=2}^{k} X_i$. Let us estimate the reliability function of the system (computer network) using the formula of full conditional probability (see [3])

$$h(X_1, X_2, \ldots, X_k) = E\phi(X_1, X_2, \ldots, X_k)$$
$$= P(\phi(X_1, X_2, \ldots, X_k) = 1) =$$
$$P(\sum_{i=1}^{k} X_i \ge 1) = P(X_1 + y \ge 1)$$

$$= P(X_1 + y \geq 1 | y = 1) \cdot P(y = 1) +$$
$$P(X_1 + y \geq 1 | y = 0) \cdot P(y = 0)$$
$$= P(X_1 \geq 0) \cdot P(y = 1) + P(X_1 \geq 1) \cdot P(y = 0) \leq$$
$$P(y = 1) + P(X_1 \geq 1) = P(y = 1)$$
$$+ P(X_1 = 1) \leq P(y \geq 1) + P(X_1 = 1) =$$
$$P(\sum_{i=2}^{k} X_i \geq 1) + P(X_1 = 1) \leq \cdots$$
$$\leq \sum_{i=1}^{k} P(X_i = 1) = \sum_{i=1}^{k} P(\sup_{0 \leq s \leq t} V_i(ns) \leq k_i).$$

Thus,

$$0 \leq h(X_1, X_2, \ldots, X_k, t) \leq \sum_{i=1}^{k} P(\sup_{0 \leq s \leq t} V_i(t) \leq k_i). \quad (3)$$

However, for $t \geq \max_{1 \leq j \leq k} \frac{k_j}{\beta_j}$ and $0 < \varepsilon < 1$ (see (2))

$$0 \leq \lim_{t \to \infty} P(\sup_{0 \leq s \leq t} V_j(t) < k_j)$$
$$\leq \lim_{t \to \infty} P(\sup_{0 \leq s \leq t} V_j(t) < (1 - \varepsilon) \cdot a(t) \cdot \hat{\sigma}_j + \beta_j \cdot t)$$
$$= P\left( \lim_{t \to \infty} \frac{\sup_{0 \leq s \leq t} V_j(t) - \beta_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon \right) \quad (4)$$
$$\leq P\left( \overline{\lim_{t \to \infty}} \frac{\sup_{0 \leq s \leq t} V_j(t) - \beta_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon \right) = 0,$$

$j = 1, 2, \ldots, k.$

Thus (see (4)),

$$\lim_{t \to \infty} P\left( \sup_{0 \leq s \leq t} V_j(t) < k_j \right) = 0, \ j = 1, 2, \ldots, k. \quad (5)$$

Consequently, $\lim_{t \to \infty} h(X_1, X_2, \ldots, X_k, t) = 0$ (see (3) and (5)).

The proof of the theorem is completed.

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