Application of Weighted Polytomous Ordering Theory in Concepts Structure Analysis of Fraction Addition for Pupils

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Abstract: Weighted polytomous ordering theory (WPOT) is suitable for polytomous data set. It considers the weight on precondition relations between elements. The method WPOT would help us to find the hierarchical structure among elements. The purpose of this paper is to discuss WPOT and utilize it in analyzing the mathematics measurement data. The research applies WPOT in the concept structures analysis of fraction addition for pupils. The results show the relationship and hierarchies among concepts. Further suggestions and recommendations are also discussed.

Key-Words: weighted polytomous ordering theory, ordering theory, fraction addition, concepts structure, cognition diagnosis.

1 Background and Motivation

In the viewpoint of learning process and psychometrics, the concept structure is an important issue which is worth discussing [9]. It is important to find the ordering relation and hierarchical structure among concepts. Ordering theory (OT) was developed by Bart and Krus and it aims to find the precondition between items based on cross-tabulation of dichotomous items [6] [7]. Bart and Krus proposed to find hierarchies among items by OT so that it is helpful to comprehend the structure among items. However, it can't deal with the polytomous data. That is, OT can only analyze the dichotomous data and this limits its application in many fields. For this reason, Lin, Bart, and Huang extended the dichotomous OT to weighted polytomous ordering theory (WPOT). Both weight of precondition and polytomous score are considered so that WPOT could extend the application of ordering theory [8]. In addition to popular application on item hierarchies, ordering theory could also be applied in the analysis of concept hierarchies. Therefore, it can describe the relation of hierarchical structures among concepts.

Many researches about OT are discussed in the past [2][3][4][5]. Lin, Bart, and Huang extended OT to WPOT and implemented the computer software. This research will adopt WOPT for concept structure analysis of fraction addition for pupils.

Concepts of fractional are important units in the mathematics course of elementary schools [10]. Related literatures show that pupils make quite a few misconceptions on fraction and calculation operation. They have many baffles on learning process of meaning and operations of fraction. Hence, in order to investigate the hierarchical structures among concepts for pupils, this research utilize WPOT to explore the cognition diagnosis about fractional addition.

2 Literature Review

The ordering theory proposed by Bart and Krus can find the ordering relation and precondition between dichotomous items by cross-tabulation [1]. For example, both item i and item j are dichotomous. Correct answer is denoted by 1 and wrong answer is denoted by 0. The cross-tabulation on number of subjects is shown as Table 1.

Table 1. Cross Table on Number of Subjects for Dichotomous Item i and Item j

		Item j 1	Item j 0	Total
Item i	1	n_{11}	n_{10}	$n_{1\bullet}$
	0	n_{01}	n_{00}	$n_{0\bullet}$
Total		$n_{\bullet 1}$	$n_{\bullet 0}$	n

In Table 1, n_{11} means the number of subjects who answer correctly on both item i and item j; n_{10} is the number of subjects who answer correctly on item i, but incorrect on item j; $n_{1\bullet}$ means the number of subjects who answer correctly in item i; $n_{0\bullet}$ is the number of subjects who give incorrect answer on item i.

According to Table 1, Bart and Krus proposed item i and item j which have four response combinations. They are (1,1) (1,0) (0,1) (0,0). It is only (0,1) that can't conform item i which is a precondition of item j [1]. Therefore, Bart and Krus defined the value n_{01}/n. The threshold ε ($0<\varepsilon<1$) is to decide whether item i is the precondition of item j. It is $0<n_{01}/n<1$. The smaller value the n_{01}/n is, the more degree item i is the precondition of item j.

The threshold ε to decide whether item i is the precondition of item j is as follows.

(1) If $(n_{01}/n)<\varepsilon$ exists, it represents item i is the precondition of item j. There is ordering relation between item i and item j and it is shown as $i \to j$.

(2) If $(n_{01}/n)\geq\varepsilon$ exists, it represents item i is not the precondition of item j. There is no ordering relation between item i and item j, and no line which links item i and item j.

3 Method of Data Analysis

WPOT is beyond the limitation of the dichotomous ordering theory. The procedure of WPOT analysis is as follow [8].

(1) Assume that the total scores of item i and item j are C_i-1 and C_j-1 respectively. The scoring is presented by $k=0,1,\cdots,(C_i-1)$ and $l=0,1,\cdots,(C_j-1)$. The cross-tabulation on numbers of subjects between item i and j is shown as Table 2, and it is $n=\sum_{k=0}^{C_i-1}\sum_{l=0}^{C_j-1}n_{kl}$.

Table 2. Cross Table on Number of Subjects for Polytomous Item i and Item j

Item i	Item j C_j-1	...	0	Total
C_i-1	$n_{(C_i-1)(C_j-1)}$	…	$n_{(C_i-1)0}$	$n_{(C_i-1)\bullet}$
C_i-2	$n_{(C_i-2)(C_j-1)}$	…	$n_{(C_i-2)0}$	$n_{(C_i-2)\bullet}$
⋮	⋮	…	⋮	⋮
1	$n_{1(C_j-1)}$	…	n_{10}	$n_{1\bullet}$
0	$n_{0(C_j-1)}$	…	n_{00}	$n_{0\bullet}$
Total	$n_{\bullet(C_j-1)}$	…	$n_{\bullet 0}$	n

(2) Define the weighted value for response combinations which unsatisfy item i is the precondition of item j. It is

$$w_{kl}=\frac{l}{C_j-1}-\frac{k}{C_i-1}\ ,\ \forall\frac{k}{C_i-1}<\frac{l}{C_j-1} \quad (1)$$

Where $0\leq k\leq(C_i-1)$, $0\leq l\leq(C_j-1)$

(3) Consider and define the weighted frequencies which unsatisfy item i is the precondition of item j. It is

$$n'=\sum_k\sum_l w_{kl}n_{kl}\quad ,\quad \forall\frac{k}{C_i-1}<\frac{l}{C_j-1} \quad (2)$$

Where $0\leq k\leq(C_i-1)$, $0\leq l\leq(C_j-1)$

(4) Define the value of n'/n as the measured coefficient of item i which directs to item j. The range of n'/n is $0\leq(n'/n)\leq 1$. The smaller value the n'/n is, the more degree item i is the precondition of item j.

(5) According to the selected threshold ε ($0<\varepsilon<1$), one can decide the ordering relation among items as follows.

- If $(n'/n)<\varepsilon$ exists, it represents that item

i is the precondition of item j. There is ordering relation between item i and item j. It is shown as $i \rightarrow j$.

- If $(n'/n) \geq \varepsilon$ exists, it means that item i is not the precondition of item j. There is no ordering relation between item i and item j, and no line which links item i and item j.

Lin, Bart and Huang implemented the WPOT software and it can apply to the data of polytomous and mixed scoring. If the scoring of item i and item j is $C_i = C_j$, it is called homogeneous scoring. On the contrary, it is called heterogeneous scoring when it is $C_i \neq C_j$. Furthermore, when it is $C_i = C_j = 2$, WPOT is reduced to dichotomous OT. Therefore, dichotomous OT is the special case of WPOT. This research would utilize WPOT to analyze an empirical data on fraction addition measurement and its scoring is heterogeneous.

4 Research Design

In this research, the instrument is fraction addition test which includes 17 items. These 17 items measure 6 concepts. The subjects are 851 sixth graders from Taiwan. The content of concepts in this test is shown as Table 3. The unit of the data analysis is concept.

Table 3. The Content of Concepts

Concepts	Content
1	Same denominator (denominator and numerator are relative prime)
2	Different denominator (denominator and numerator are multiple)
3	Different denominator (denominator and numerator are not multiple but have a common factor)
4	Different denominator (denominators are relatively prime)
5	Different denominator (denominators are multiple)
6	Different denominator (denominators aren’t multiple but have a common factor)

Each item will measure some of the above concepts. The relation between items and concepts are depicted in Table 4. Table 4 is called concept attributes matrix of items. In the matrix, the value 1 means the item exactly measure the corresponding concept. For instance, item 2 only measures concept 5 and item 3 measures concept 2 and concept 4. There are total score for each concept. For example, concept 1 has total score 4. Therefore, it is $C_1 - 1 = 4$ and $0 \leq k \leq (C_1 - 1)$ in WPOT. As shown in Table 4, each concept has varied total score and therefore its scoring is heterogeneous.

Table 4. The Concept Attributes Matrix of Items

Items	Concepts					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	0	0	0	1	0
3	0	1	0	1	0	0
4	0	0	0	0	0	1
5	0	0	0	1	0	0
6	1	1	0	0	0	1
7	1	0	0	0	0	0
8	0	0	0	0	1	0
9	0	0	0	0	0	1
10	1	1	0	0	0	1
11	0	0	1	1	0	0
12	0	0	0	1	0	0
13	0	0	0	0	1	0
14	0	0	0	0	0	1
15	0	0	0	1	0	0
16	0	0	0	0	0	1
17	0	0	0	0	0	1
Total Score	4	3	1	5	3	7

The contents of all fraction addition items are depicted in Table 5. There are 17 items in the fraction addition test. This test is a paper-pencil test and it takes about 30 minutes for most pupils to finish it. According to the related literatures, the concepts in Table 3 will influence the performance of test for pupils [9] [10]. Especially, there will be some misconceptions on fraction for pupils of learning deficiency in mathematics [11]. Therefore, the application of WPOT on cognition diagnosis of fraction addition will be feasible. The results should provide meaningful information for pedagogy and remedial instruction.

Table 5. The Contents of Items

Item	Content	Item	Content
1	$\frac{1}{5}+\frac{3}{5}$	10	$\frac{7}{14}+\frac{4}{8}$
2	$\frac{1}{3}+\frac{1}{6}$	11	$\frac{15}{21}+\frac{6}{8}$
3	$\frac{3}{8}+\frac{3}{9}$	12	$\frac{1}{2}+\frac{1}{11}$
4	$\frac{1}{6}+\frac{2}{9}$	13	$\frac{3}{13}+\frac{1}{26}$
5	$\frac{6}{7}+\frac{2}{3}$	14	$\frac{1}{12}+\frac{5}{18}$
6	$\frac{2}{4}+\frac{3}{6}$	15	$\frac{3}{11}+\frac{2}{13}$
7	$\frac{2}{13}+\frac{5}{13}$	16	$\frac{5}{12}+\frac{4}{15}$
8	$\frac{2}{9}+\frac{1}{27}$	17	$\frac{1}{21}+\frac{3}{14}$
9	$\frac{1}{6}+\frac{2}{15}$		

This research is to analyze concepts structures of fraction addition by WPOT. We have two known matrix and they are as follows.

(1) $D=(d_{nj})_{851\times 17}$ denotes the response data matrix of all subjects. If subject n give correct answer on item j, it is $d_{nj}=1$; otherwise, it is $d_{nj}=0$.

(2) $A=(a_{jk})_{17\times 6}$ is the concept attributes matrix in Table 4. If item j exactly measure concept k, it is $a_{jk}=1$; $a_{jk}=0$ means item j does not measure concept k.

With these two matrices D and A, the scoring of concepts are defined as $S=(s_{nk})_{851\times 6}$, where s_{nk} means the score of subject n on concept k. It is

$$S=(D)(A)=(s_{nk})_{851\times 6} \qquad (3)$$

Threshold value $\varepsilon=0.2$ is selected in this study. With the analysis of WPOT, the output figure will reveal the concepts structure of fraction addition for pupils.

5 Result

According to the analysis of WPOT, the weighted percentages n'/n which unsatisfy item i is the precondition of item j is depicted in Table 6. With the selected threshold $\varepsilon=0.2$, the ordering relation between concepts is shown in Table 7.

Table 6. Weighted Percentages n'/n of Concepts

Concepts	1	2	3	4	5	6
1	---	.01	.02	.01	.05	.01
2	.16	---	.04	.03	.10	.03
3	.43	.30	---	.20	.34	.26
4	.26	.12	.04	---	.17	.09
5	.15	.06	.03	.03	---	.04
6	.21	.08	.05	.04	.13	---

Table 7. Ordering Relation between Concepts

Concepts	1	2	3	4	5	6
1	---	1	1	1	1	1
2	1	---	1	1	1	1
3	0	0	---	0	0	0
4	0	1	1	---	1	1
5	1	1	1	1	---	1
6	0	1	1	1	1	---

The hierarchical structure of fraction addition is depicted in Fig. 1. There are three levels for this concept structure. One is concluded that concept 1, 2, and 5 are the preconditions of concept 4 and 6. Furthermore, concept 4 and 6 are the preconditions of concept 3. It shows the hierarchies and relationship among concepts. With of this concept structure, its information could provide useful references for mathematics education.

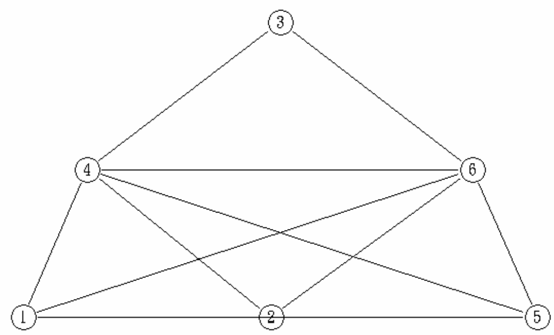

Fig 1. WPOT Graph of Concept Structure

6 Conclusions

The result shows the hierarchical structure of concepts about fraction addition based the WPOT analysis method. The information could help teachers to adopt remedial teaching in the educational environment.

Further studies could extend the application of WPOT to other learning field, like nature science or skill learning. On the other hand, transitivity on ordering theory is an important issue. How to maintain the property of transitivity on WPOT will be a prospective research.

References:

[1] P. W. Airasian, and W. M. Bart, Ordering theory: A New and Useful Measurement Model, *Educational Technology*, 1973, pp. 56-60.

[2] P. W. Airasian, W. M . Bart, and B. J. Greaney, The Analysis of a Propositional Logic Game by Ordering Theory, *Child Study Journal*, Vol. 5, 1975, pp. 13-24.

[3] W. M. Bart, Some Results of Ordering Theory for Guttman Scaling, *Educational and Psychological Measurement*, Vol. 36, 1976, pp. 141-148.

[4] W. M. Bart, and P. W. Airasian, Determination of the Ordering among Seven Piagetian Tasks by an Ordering-theoretic Method, *Journal of Educational Psychology*, Vol. 66, 1974, pp. 277-284.

[5] W. M. Bart, S. Frey, and J. Baxter, Generalizability of the Ordering among Five Formal Reasoning Tasks by an Ordering-theoretic Method, *Child Study Journal*, Vol. 9, 1979, pp. 251-259.

[6] W. M. Bart, and D. J. Krus, An Ordering-theoretic Method to Determine Hierarchies among Items, *Educational and Psychological Measurement*, Vol. 33, 1973, pp. 291-300.

[7] W. M. Bart, and S. A. Read, A Statistical Test for Prerequisite Relations, *Educational and Psychological Measurement*, Vol. 44, 1984, pp. 223-227.

[8] Y. H. Lin, W. M. Bart, and K. J. Huang, *WPOT software* [manual and software for generalized scoring ordering theory]. Taiwan, Taichung City: National Taichung University, 2006.

[9] R. J. Mislevy, and N. Verhelst, Modeling Item Response When Different Subjects Employ Different Solution Strategies, *Psychometrika*, Vol. 55, 1990, pp. 195-215.

[10] K. K. Tatsuoka, and M. M. Tatsuoka, Detection of Aberrant Response Patterns, *Journal of Educational Statistics*, Vol. 7, 1982, pp. 215-231.

[11] K. K. Tatsuoka, and M. M. Tatsuoka, Computerized Cognitive Diagnostic Adaptive Testing: Effect on Remedial Instruction as Empirical Validation, *Journal of Educational Measurement*, Vol. 34, 1997, pp. 3-20.