Mixed Scoring on Polytomous IRS and the Application in Concepts Diagnosis for Fraction Subtraction

MIN-YEN CHEN
Graduate Institute of Educational Measurement and Statistics
National Taichung University
140 Min-Shen Rd., Taichung City 403, Taiwan
Taiwan

YUAN-HORNG LIN
Department of Mathematics Education
National Taichung University
140 Min-Shen Rd., Taichung City 403, Taiwan
Taiwan

Abstract: The purpose of this study is to provide the mixed scoring of polytomous item relational structure (PIRS) analysis and utilize it in fraction subtraction concepts diagnosis. There are limitations on dichotomous item relational structure (IRS) and PIRS analysis is well efficient in the analysis of polytomous data. Most utilization of IRS aims to analyze the hierarchical structures of items, not the knowledge structures of conceptual attributes. Besides, concept diagnosis will provide important information for pedagogy. In this study, PIRS will be used in the analysis of conceptual attributes of fraction subtraction. The results show that the fraction subtraction learning concepts diagnosis by the method of PIRS analysis is feasible. Finally, some suggestions and recommendations are discussed.

Key-Words: polytomous item relational structure, fraction subtraction, concept diagnosis, knowledge structure, educational measurement.

1 Background and Motivation

Based on ordering theory which was developed by Airasian, Bart and Krus [1][2][3], M. Takeya proposed another formula of ordering coefficient which was called item relational structure (IRS) [8]. However, there is limitation in its scoring because IRS is suitable for dichotomous items. As to the testing items, there are a variety of item formats, such as multiple-choice items, true-false items, matching items, completion items, and computation items. Some of these varied items are polytomous. Therefore, polytomous items are essential in real situation, but IRS could not be used for polytomous items.

According to the deficiency of IRS, Lin, Bart and Huang proposed an advanced IRS ordering coefficient formula and this formula is suitable for polytomous items, which is called polytomous item relational structure (PIRS). As to this improvement, dichotomous IRS is a special case of PIRS [17]. Besides, PIRS is also suitable for mixed scoring situations, which is not a homogeneous scoring on items. For example, the total score of item 1 is 3, but the total score of item 2 is 5. Therefore, PIRS could process polytomous data with well efficiency and extend the application of ordering theory in real assessment environment.

Most research related to dichotomous IRS is to analyze the item hierarchy, not concept hierarchy. On the other hand, concept hierarchy analysis is also an important issue but little is known related to the application of IRS on concept hierarchy. With the analysis on concept hierarchy, its results could provide information for cognition diagnosis and pedagogy.

As to the mathematics education in elementary schools, the concepts of fraction subtraction are important for pupils in mathematics learning [6]. Based on findings of existing research, students often make incorrect computation in the process of fraction subtraction learning and result in far and deep effect on relative concepts learning in the future such as fraction multiplication and fraction division [13]. There are few studies on fraction subtraction concepts diagnosis based on the attributes of fraction subtraction [14].

In the related research of fraction concepts development, it shows that the learning of fraction concepts is influenced by learning process [5]. These factors include number concept, the difference of proper fraction and mixed fraction and the representation of characters and figures. Quite a few research show that students deal with proper fraction much easier than improper fraction. Also, students deal with same denominator easier than different denominator [7] [12]. However, the knowledge structure of fraction subtraction is not clearly understood. Therefore, cognition diagnosis on fraction subtraction should be prospective and substantial for mathematics education.

In all, this study is to utilize PIRS analysis for pupils on fraction subtraction concepts diagnosis. In addition, owing to the polytomous scoring on fraction attributes in the empirical study, PIRS will be applied to analyze the concept hierarchy. The results of this empirical analysis could provide references and suggestions for mathematics education.

2 Literature Review

Calculation of dichotomous IRS and an example is discussed in the following literature review.

2.1 Dichotomous Item Relational Structure

M. Takeya proposed the method of ordering coefficient which was called item relational structure (IRS) and was used in the analysis of dichotomous data [8] [9] [11] . IRS is another branch of ordering theory, developed by Airasian, Bart and Krus [4]. Based on the cross table of response data, computation on the subordinate relation will decide the precondition and ordering relationship between items. The calculation on ordering coefficient of IRS is depicted as follows.

To take dichotomous item for example, item i and item j are designed in dichotomous format, correct answer is recorded by “1”and wrong answer is recorded by “0”. The cross table of ratio for all examinee is shown as Table 1.

Table 1. Cross Table for Ratio of Examinee between Item i and Item j

		Item j 1	Item j 0	Total
Item i	1	p_{11}	p_{10}	$p_{1\bullet}$
	0	p_{01}	p_{00}	$p_{0\bullet}$
Total		$p_{\bullet 1}$	$p_{\bullet 0}$	1

According to Table 1, it is obvious that $p_{11} + p_{10} + p_{01} + p_{00} = 1$. M. Takeya defined ordering coefficient to represent the degree of item i as a precondition of item j [10]. It is as follows.

$$r_{ij}^{*} = 1 - \frac{p_{01}}{(p_{\bullet 1})(p_{0\bullet})} \tag{1}$$

In the above formula, p_{01} is the joint ratio of examinee who are incorrect on item i but correct on item j; $p_{\bullet 1}$ is the marginal ratio of examinee who are correct on item j; $p_{0\bullet}$ is the marginal ratio of examinee who are incorrect on item i.

The greater the r_{ij}^{*} is, the greater degree the item i will be the precondition of item j. Thus, M. Takeya proposed the threshold ε so that the coefficient r_{ij} could decide the precondition between items. It is

$$r_{ij} = \begin{cases} 1 & , \quad r_{ij}^{*} \ge \varepsilon \\ 0 & , \quad r_{ij}^{*} < \varepsilon \end{cases} \tag{2}$$

If $r_{ij} = 1$ exists, it means item i is the precondition of item j and it is shown as $i \rightarrow j$ in the item hierarchy graph. Otherwise, $r_{ij} = 0$ means item i is not the precondition of item j and there is no direction from i to j. M. Takeya suggested ε value could be decided by $\varepsilon = .50$.

2.2 Example of Dichotomous Item Relational Structure Analysis

An example of response data with 6 dichotomous items and 10 subjects is shown as Table 2.

Table 2. An Example of Response Data with 6 Dichotomous Items and 10 Subjects

Subjects	Items 1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	0	0
4	1	1	1	0	1	0
5	1	1	1	0	0	0
6	0	1	1	0	1	0
7	1	0	0	1	0	0
8	1	0	0	1	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
Correct Ration	.70	.60	.60	.50	.40	.20

According to the dichotomous IRS analysis, the ordering coefficients r_{ij}^* among these dichotomous items are shown in Table 3. With $\varepsilon = .50$, the coefficients r_{ij} of these dichotomous items are shown in Table 4. Finally, the item hierarchy graph of this example is depicted in Fig. 1. As shown in Fig. 1., the levels and preconditions among items with their correct ratio display their hierarchical relations.

Table 3. The Ordering Coefficient r_{ij}^* of 6 Dichotomous Items

Items	Items					
	1	2	3	4	5	6
1	----	.44	.44	1.00	.17	1.00
2	.29	----	1.00	.00	1.00	1.00
3	.29	1.00	----	.00	1.00	1.00
4	.43	.00	.00	----	.00	1.00
5	.05	.44	.44	.00	----	1.00
6	.11	.17	.17	.25	.38	----

Table 4. The Coefficient r_{ij} of 6 Dichotomous Items with Threshold .50

Items	Items					
	1	2	3	4	5	6
1	----	0	0	1	0	1
2	0	----	1	0	1	1
3	0	1	----	0	1	1
4	0	0	0	----	0	1
5	0	0	0	0	----	1
6	0	0	0	0	0	----

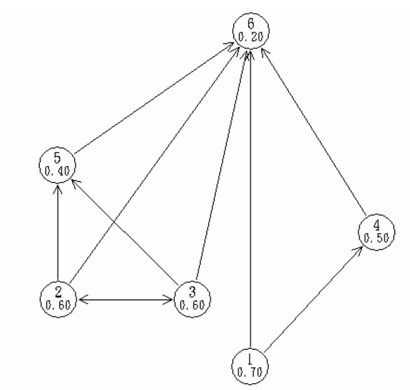

Fig. 1. The Item Hierarchy of 6 Items

3 Calculation of Polytomous IRS

Owing to the limitation of dichotomous IRS, Lin, Bart and Huang provided polytomous item relational structure (PIRS) and implemented the software [17]. The calculation of PIRS is described as following steps.

(1) Assume the counting scores of item i and item j are C_i and C_j categories respectively. That is, the score of item i is denoted by k, where $k = 0,1,\cdots,(C_i - 1)$. Similarly, the score of item j is denoted by l, where $l = 0,1,\cdots,(C_j - 1)$. The cross table of ratio for all examinee is shown as Table 5.

Table 5. Cross Table for Ratio of Examinee between Item i and Item j

Item i	Item j			Total
	$C_j - 1$	$\cdots$	0	
$C_i - 1$	$p_{(C_i-1)(C_j-1)}$	$\cdots$	$p_{(C_i-1)0}$	$p_{(C_i-1)\bullet}$
$C_i - 2$	$p_{(C_i-2)(C_j-1)}$	$\cdots$	$p_{(C_i-2)0}$	$p_{(C_i-2)\bullet}$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
1	$p_{1(C_j-1)}$	$\cdots$	p_{10}	$p_{1\bullet}$
0	$p_{0(C_j-1)}$	$\cdots$	p_{00}	$p_{0\bullet}$
Total	$p_{\bullet(C_j-1)}$	$\cdots$	$p_{\bullet 0}$	1

In Table 5, p_{kl} is the joint ratio of examinee who get score k on item i and score l on item j; $p_{k\bullet}$ is the marginal ratio of examinee who get score k on item i; $p_{\bullet l}$ is the marginal ratio of examinee who get score l on item j. With the notation in Table 5, it is obvious that $\sum_{k=0}^{C_i-1}\sum_{l=0}^{C_j-1} p_{kl} = 1$.

(2) Define a set of item matches which unsatisfy "item i is the precondition of item j". Thus, the definition of this set is as follows.

$$A = \left\{ (k,l) \middle| \frac{k}{C_i - 1} < \frac{l}{C_j - 1} \right\} \quad (3)$$

(3) The number of elements in set A is $\# A$. With consideration on standardization, the ordering coefficient R_{ij}^* to represent the degree of item i as a precondition of item j is as follows.

$$R_{ij}^* = 1 - (\frac{1}{\# A}) \sum_k \sum_l \frac{p_{kl}}{(p_{\bullet k})(p_{l\bullet})}\ , \ \forall \frac{k}{C_i - 1} < \frac{l}{C_j - 1} \quad (4)$$

(4) The threshold ε is chosen so that the coefficient R_{ij} could decide the precondition between items.

It is

$$R_{ij} = \begin{cases} 1 & , \quad R_{ij}^* \geq \varepsilon \\ 0 & , \quad R_{ij}^* < \varepsilon \end{cases} \qquad (5)$$

If $R_{ij} = 1$ exists, it means item i is the precondition of item j and it is shown as $i \rightarrow j$ in the item hierarchy graph. On the contrary, $R_{ij} = 0$ means item i is not the precondition of item j and there is no direction from i to j.

If it is dichotomous for both items, it will be $C_i = C_j = 2$ and the PIRS calculation is the same with dichotomous IRS. Therefore, dichotomous IRS is a special case of PIRS.

4 Research Design and Analysis Process

There are 852 valid subjects of sixth graders from Taiwan. These subjects are about 11 years old. The test instrument is the fraction subtraction test which includes 11 items. Based on the related literature on fraction subtraction, there are 10 important concepts for fraction subtraction [15] [16]. The test of 11 items measures these 10 concepts which are depicted in Table 6.

Table 6. The Content of Fraction Subtraction Concepts

Concepts	Content
1	Judgment for same denominator
2	Judgment for different denominator
3	Minuend is mixed fraction
4	Subtrahend is mixed fraction
5	Minuend can be reduced
6	Subtrahend can be reduced
7	Minuend abdication
8	Find a common denominator (Denominators are multiple)
9	Find a common denominator (Denominators are not multiple but have a common factor)
10	Find a common denominator (Denominators are relatively prime)

With regard to these 10 concepts, the concept attributes matrix of the test is shown in Table 7. If the value is 1, it means the item exactly measures the corresponding concept; otherwise, it doesn't measure the corresponding concept. For example, item 2 measures concept 1, 3, 6, 7.

Table 7. The Concept Attributes Matrix of the Test

Item	Concepts									
	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	1	1	0	0	0
3	1	0	1	1	0	1	1	0	0	0
4	0	1	0	0	0	0	0	1	0	0
5	0	1	1	0	0	0	0	1	0	0
6	0	1	1	1	1	0	1	1	0	0
7	0	1	0	0	1	1	0	0	0	1
8	0	1	0	0	0	0	0	0	0	1
9	0	1	0	0	0	0	0	0	1	0
10	0	1	0	0	1	0	0	0	1	0
11	0	1	1	0	0	0	1	0	1	0
Total	3	8	5	2	3	3	4	3	3	2

Further, the contents of fraction subtraction items of the test are shown as Table 8.

Table 8. The Contents of Fraction Subtraction Test

Item	Content	Item	Content
1	$\frac{7}{11}-\frac{5}{11}$	7	$\frac{6}{8}-\frac{6}{9}$
2	$5\frac{3}{8}-\frac{6}{8}$	8	$\frac{5}{6}-\frac{2}{5}$
3	$7\frac{1}{6}-3\frac{2}{6}$	9	$\frac{7}{12}-\frac{3}{8}$
4	$\frac{5}{9}-\frac{7}{18}$	10	$\frac{10}{16}-\frac{5}{12}$
5	$5\frac{3}{4}-\frac{3}{8}$	11	$3\frac{1}{8}-\frac{5}{6}$
6	$7\frac{8}{12}-3\frac{5}{6}$		

This research aims to analyze concepts structures of fraction subtraction by PIRS. We have two known matrix and they are as follows.

(1) $D = (d_{nj})_{852\times 11}$ denotes the response data matrix of all subjects. $d_{nj} = 1$ means subject n give correct answer on item j; otherwise, $d_{nj} = 0$ means subject n give wrong answer on item j.

(2) $A = (a_{jk})_{11\times 10}$ denotes the concept attributes matrix in Table 7. $a_{jk} = 1$ means item j exactly measure concept k; otherwise, $a_{jk} = 0$ means item j does not measure concept k.

With these two matrices D and A above, the scoring of concepts are defined as follows. s_{nk} means the score of subject n on concept k.

$$S = (D)(A) = (s_{nk})_{852\times 10} \qquad (6)$$

To take concept 1 for example, its total score is 3 in Table 7. Therefore, there will be 4 different scoring categories and they are 0, 1, 2, 3 respectively. It is denoted $C_1 = 4$. The matrix $S = (D)(A)$ will be source data for PIRS analysis in this study.

5 Results

Based on analysis of PIRS, the ordering coefficients R^*_{ij} among concepts are acquired and they are depicted in Table 9.

Table 9. The Ordering Coefficients R^*_{ij} between Concepts.

Concepts	Concepts									
	1	2	3	4	5	6	7	8	9	10
1	--	.68	.89	.87	.81	.99	.91	.72	.76	.65
2	-.19	--	.09	-.02	.40	-.19	.09	-.28	.49	-.05
3	-.43	.28	--	.23	.45	-.25	.20	.11	.44	.18
4	-.38	.32	.26	--	.50	.02	.45	.27	.42	.37
5	-.25	-.24	-.10	.04	--	-.08	-.03	.06	.46	.21
6	-.47	.43	..63	.56	.79	--	.70	.53	.59	.49
7	-.44	.34	.29	.45	.44	-.25	--	.13	.46	.21
8	-.20	.31	.20	.28	.70	.10	.24	--	.51	.20
9	-.17	-.37	-.13	-.07	.19	-.16	-.09	-.06	--	-.14
10	-.18	.21	.07	.22	.47	-.03	.13	.15	.44	--

The threshold $\varepsilon = .60$ is chosen in this empirical data analysis [17]. The prerequisite relations between concepts are depicted in Table 10. That is, if the value between two concepts satisfies $R^*_{ij} \geq .60$ in Table 9, the corresponding prerequisite relation between two concepts in Table 10 is recorded as 1 and there is linkage for these two concepts; otherwise, the prerequisite relation between two concepts in Table 10 is recorded as 0 and there is no linkage between two concepts.

According to the prerequisite relations between concepts in Table 10, the concepts hierarchies and relations are depicted in Fig. 2.

Table 10. The Ordering Coefficients R_{ij} between Concepts.

Concepts	Concepts									
	1	2	3	4	5	6	7	8	9	10
1	--	1	1	1	1	1	1	1	1	1
2	0	--	0	0	0	0	0	0	0	0
3	0	0	--	0	0	0	0	0	0	0
4	0	0	0	--	0	0	0	0	0	0
5	0	0	0		--	0	0	0	0	0
6	0	0	1	0	1	--	1	0	0	0
7	0	0	0	0	0	0	--	0	0	0
8	0	0	0	0	1	0	0	--	0	0
9	0	0	0	0	0	0	0	0	--	0
10	0	0	0	0	0	0	0	0	0	--

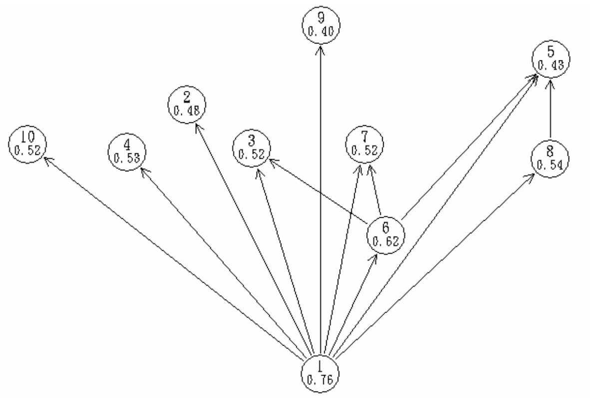

Fig. 2. The Concept Hierarchies of Fraction Subtraction

As shown in Fig. 2., there are relations and hierarchies among concepts. The correct ration is displayed within the circle of each concept. For example, the correct ration of concept 1 is .76. Some following findings could be concluded based on Fig. 2.

(1) Concept 1 (judgment for same denominator) is the precondition of all the other concepts (concept 2, 3, 4, 5, 6, 7, 8, 9, 10). It is because there is direct direction from concept 1 to all these concepts. This shows that "judgment for same denominator" is easier to learn for sixth graders (11 years old) and it is the basic skill and foundation of all related fraction subtraction concepts.

(2) Concept 6 (subtrahend can be reduced) is the precondition of concept 3 , 5 , 7. This means that "subtrahend can be reduced" is the prerequisite learning elements of concept 3 , 5 , 7.

(3) Concept 8 (find a common denominator and denominators are multiple) is the precondition of concept 5 and concept 1 is the precondition of

concept 8. It means concept 8 is also located in the mediate relation between concept 1 and concept 5.

6 Conclusions

PIRS is beyond the limitation of dichotomous scoring data. This research adopts PIRS in the analysis of fraction subtraction concepts. Unlike the traditional IRS method on analysis of items, PIRS is extended to analyze concepts in this study. This research also shows that polytomous data on concept measurement can be well analyzed under PIRS method.

The results of this empirical data could provide information for cognition diagnosis and remedial instruction in educational environment. To sum up, this method could provide an alternative approach of cognitive diagnosis methodology.

References:

[1] P. W. Airasian, and W. M. Bart, Ordering theory: A New and Useful Measurement Model, *Educational Technology*, May, 1973, pp. 56-60.

[2] W. M. Bart, and D. J. Krus, An Ordering-theoretic Method to Determine Hierarchies among Items, *Educational and Psychological Measurement*, Vol. 33, 1973, pp. 291-300.

[3] W. M. Bart, Some Results of Ordering Theory for Guttman Scaling, *Educational and Psychological Measurement*, Vol. 36, 1976, pp. 141-148.

[4] W. M. Bart, and S. A. Read, A Statistical Test for Prerequisite Relations, *Educational and Psychological Measurement*, Vol. 44, 1984, pp. 223-227.

[5] M. A. Clements, and G. A. Lean, Discrete Fraction Concepts and Cognitive Structure, *P.M.E.*, Vol. XII, 1987, pp. 215-232.

[6] J. Hiebert, and L. H. Tonnessen, Development of the Fraction Concept in two Physical Context: an Exploratory Investigation, *Journal for Research in Mathematics Education*, Vol. 9, 1978, pp. 374-378.

[7] R. Mislevy, Evidence and Inference in Educational Assessment, CSE Technical Report 414, 1996.

[8] M. Takeya, Construction and Utilization of Item Relational Structure Graphs for Use in Test Analysis. *Japan Journal of Educational Technology*, Vol. *5*, 1980, pp. 93-103.

[9] M. Takeya, *New Test Theory∶Structural Analysis*, Tokyo：Waseda University Press. 1991.

[10] M. Takeya, and H. Sasaki, An Evaluation Method for Students' Understanding Using Their Cognitive Maps, *Transactions of Institute of Electronics, Information and Communication Engineer*s, J80-D-II, Vol. 1, 1997, pp. 336-347.

[11] M. Takeya, *Structure Analysis Methods for Instruction*, Tokyo: Takushoku University Press, 1999.

[12] K. Tatsuoka, Rule Space: An Approach for Dealing With Misconceptions based on Item Response Theory, *Journal of Educational Measurement*, Vol. 20, 1983, pp. 345-354.

[13] K. Tatsuoka, Analysis of Errors in Fraction Addition and Subtraction Problems, *NIE Final Report. Urbana*, IL: University of Illinois. 1984

[14] M. Tatsuoka, and K. Tatsuoka, Rule Space. In S. Kotz & N. L. Johnson (Eds.), *Encyclopedia of Statistical Sciences*, New York: Wiley, 1989

[15] I. R. Katz, M. E. Martinez, K. M. Sheehan, and K. Tatsuoka, Extending the Rule Space Model to a Semantically-Rich Domain: Diagnostic Assessment in Architecture, *Journal of Educational and Behavioral Statistics*, Vol. 23, 1998, pp. 254-278.

[16] K. Tatsuoka, Toward an Integration of Item Response Theory and Cognitive Error Diagnosis. In N. Fredrickson, R. L. Glaser, A. M. Lesgold, and M. G. Shafto (Eds.), *Diagnostic monitoring of skills and knowledge acquisition*, 1990, pp. 453-488.

[17] Y. H. Lin, W. M. Bart and K. J. Huang, *WPIRS software* [manual and software for generalized scoring of item relational structure]. Taiwan, Taichung City: National Taichung University, 2006.