An Extension of Ordering Theory on Scoring and Its Application in Cognition Diagnosis of Capacity Concepts

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Abstract: Ordering theory method was used to determine existing hierarchies within a set of items. But the method is restricted to dichotomously scored items. This limitations narrow its application in many empirical data. The purpose of this study is to introduce polytomous ordering theory (POT), developed by Y. H. Lin , W. M. Bart and K. J. Huang, and apply it in the knowledge structure analysis of capacity concepts. It can be used to analyze data set not only of dichotomous items but also of polytomous items. In addition to item hierarchies, the methodology of concepts hierarchies is also an important issue. Therefore, the authors apply POT in the investigation capacity concepts for pupils. The results of data analysis will provide references for the remedial teaching and the design of teaching materials.

Key-Words: capacity concepts, knowledge structure, ordering theory, polytomous items

1 Background and Motivation

The ordering relationship among elements in a complex system is an important issue in many research fields. With the benefit of ordering analysis, the hierarchies and relationship could be clearly understood. Quite a few methodologies, like interpretive structural modeling, fuzzy ordering and fuzzy structural modeling, are useful method in management and engineering [13] [15] [16].

As to psychological and educational measurement, the investigation of ordering and hierarchies between items or concepts are useful in education environment [1] [17]. It is because this information could supply the remedial instruction and cognition diagnosis. Therefore, [2] provided ordering theory (OT) for dichotomous scoring items. This dichotomous method analysis could be used to identify a hierarchical organization among items. And ordering theory has its primary intent either the testing of hypothesized hierarchies among items or the determination of hierarchies among items. With the analysis of OT, the test data can be analyzed so that prescriptive and diagnostic information can be provided for teachers and researchers [2] [3]. Besides, OT is usually used as cognitive diagnosis in education and cognitive development stage validation.

OT has been greatly discussed for a long time. It was also extended to multidimensional scaling of dichotomous items and other utility in intelligence test [4][5][18]. However, it is limited to dichotomous scoring items. This limit probably caused an inconvenience when we want to analyze polytomous scoring items. Therefore, [12] extended the dichotomous OT to polytomous ordering theory (POT). This improvement should give extension on application of ordering theory for teachers, test practitioners, and researchers. Consequently, this paper will present the foundation of POT and applies this method in the cognition diagnosis of pupils. POT method will be used to find ordering and hierarchies for capacity concepts.

As to capacity concepts, "Quantity and Measure" plays an important role in mathematics curriculum of elementary school. However, some literatures show that there exist misconceptions as to capacity concepts [8] [10]. In this study the scoring of capacity concepts is polytomous. Therefore, it is feasible and necessary to investigate the capacity concepts based on POT analysis. Generally speaking, for pupils of elementary school, "capacity" and "volume" are introduced at the same time. Both these two terms are often used interchangeably in elementary classroom [6]. On the contrary, volume and capacity are distinguished in Grade 1-9 mathematics curriculum in Taiwan and they are introduced in varied grade. Besides, most of the related studies for "Quantity and Measure" focused on the concepts of area, length and weight [22]. Little is known as to the concepts of capacity and volume. Moreover, most research for concepts of capacity and volume just use descriptive statistics to explain the data set, although their paper-pencil tests spend a lot of time [26]. Therefore, the information about concept hierarchies and ordering of concepts are limited [28].

Based on the above discussions, the POT will be used to investigate the concepts hierarchies. We designed the "capacity and volume" test tool for fifth graders and these concepts are polytomous scoring. We would also analyze data set with POT so that concept hierarchies will display the knowledge structures of pupils.

2 Literature Review

2.1 Dichotomous Ordering theory

[2] [4] furnished the dichotomous OT method for dichotomously scored items. OT is mainly used to determinate the ordering relationship of precondition between two items in psychometrics studies [3]. Item hierarchy can be displayed by the analysis of OT. To Take item i and item j ($i \neq j$), which are both dichotomous scoring, for an example, right answer is represented by 1 and wrong is 0. Four response patterns, which are (1,1), (1,0), (0,1), (0,0) are considered. The response pattern (0,1), called disconfirmatory pattern, doesn't satisfy the condition that item i is a precondition of item j [7].

A cross-table for number of examinee based on the above four response pattern (1,1), (1,0), (0,1), (0,0) could be presented in Table 1 According to Table 1, the percentage of disconfirmatory pattern (0, 1) d is defined as follows.

$$d = n_{01}/n, \text{ where } \quad 0 \leq (n_{01}/n) \leq 1 \qquad (1)$$

n_{01}/n is the percentage of disconfirmatory pattern (0, 1). The smallest the n_{01}/n is, the more probability item i is a precondition of item j [19]. Whether item i is a precondition of item j depends on tolerance level ε ($0 < \varepsilon < 1$). The hierarchical relation is defined as follows.

- If $n_{01}/n < \varepsilon$ exists, it means item i is a precondition of item j. Then, item i could be linked forward to item j ($i \rightarrow j$). And item j belong to one higher level than item i.
- If $n_{01}/n \geq \varepsilon$ exists, it means item i is not a precondition of item j. Then, there is no relationship between these two items.

[2] [7] suggested that ε can be appropriately decided at 0.2. Researchers can also assign the ε in empirical studies by themselves.

Table 1. Cross Table of Examinee for Item i and Item j

		Item j		Total
		1	0	
Item i	1	n_{11}	n_{10}	$n_{1\bullet}$
	0	n_{01}	n_{00}	$n_{0\bullet}$
Total		$n_{\bullet 1}$	$n_{\bullet 0}$	$n = n_{11} + n_{10} + n_{01} + n_{00}$

2.2 Capacity Concepts and Its Related Studies

National Council of Teachers of Mathematics established some standards for teachers and instrument design [9]. Although capacity concepts belongs to the important domain of quantity and measure, it shows that capacity concepts varies across different demographic background [8]. [20]

considered that the standard of the National Council of Teachers of Mathematics (NCTM, 2000), which expects children to understand units of measurement by second graders, is unrealistic. Moreover, better principles of teaching are also suggested to encourage children to think logically for capacity [8] [9]. According to the cognition development stage theory J. Piaget, when students recognize conservation, they can compare the capacity of different containers [10]. On the other hand, some studies claimed that many children in grade 11 could understand conservation and compensation of liquid [11].

Quite a few studies investigated the cognitive development and misconceptions of capacity concepts [29]. One common point is that prior knowledge and verbal usage play an important role in the construction and change of capacity concepts [23] [27]. The other one factor which will influence the storage of capacity is the instruction and teaching in classrooms [21] [25]. The capacity concepts will also influence the knowledge formation of nature science [24].

3 Method of POT

[12] provided generalized polytomous ordering theory. This theory overcomes some shortcomings the dichotomous ordering theory. POT is one of the features in generalized polytomous ordering theory. The steps of POT analysis are described as follows.

(1) Let items i and item j have C_i and C_j categories of scoring respectively, where $k = 0,1...,(C_i - 1)$ and $l = 0,1...,(C_j - 1)$. A cross-table for number of examinee based on these $C_i \times C_j$ response patterns could be displayed as Table 2.

(2) As to Table 2, for those response patterns with $\frac{k}{C_i - 1} < \frac{l}{C_j - 1}$, they don't satisfy the condition that item i is a precondition of item j and thus they are disconfirmatory patterns.

(3) Owing to the different total score for items i and item j, the normalized method for counting the frequencies of "item i is not the prerequisite of item j" is defined as follows.

$$n' = \sum_k \sum_l n_{kl} \ , \forall \frac{k}{C_i - 1} < \frac{l}{C_j - 1} \qquad (2)$$

(4) The percentage of disconfirmatory patterns D is defined as follows.

$$D = n'/n \ \text{, where} \quad n'/n \in [0,1] \qquad (3)$$

n'/n indicate the measurement that item i is precondition of item j. The smaller the n'/n is, the higher that item i is the precondition of item j will be.

(5) Whether item i is a precondition of item j depends on tolerance level ε $(0 < \varepsilon < 1)$. The relation is defined as follows.

- If $n'/n < \varepsilon$ exists, it means item i is a precondition of item j. Then, item i could be linked forward to item j ($i \to j$). And item j belong to one higher level than item i.
- If $n'/n \geq \varepsilon$ exists, item i is not a precondition of item j. Then, there is no relationship between these two items.

It is obvious that dichotomous ordering theory , which is $C_i = C_j = 2$, is a special case of POT.

Table 2. Cross Table of Examinee for Item i and Item j with Polytomous Scoring

		Item j			Total
		$C_j - 1$	…	0	
Item i	$C_i - 1$	$n_{(C_i-1)(C_j-1)}$	…	$n_{(C_i-1)0}$	$n_{(C_i-1)\bullet}$
	⋮	⋮	…	⋮	⋮
	1	$n_{1(C_j-1)}$	…	n_{10}	$n_{1\bullet}$
	0	$n_{0(C_j-1)}$	…	n_{00}	$n_{0\bullet}$
Total		$n_{\bullet(C_j-1)}$	…	$n_{\bullet 0}$	n

4 Research Design

The capacity concepts test is designed for fifth graders. There are 883 valid subjects of fifth graders (10 years old) from 23 elementary schools in Taiwan. Based on the mathematics curriculum standard of Taiwan, there are four core capacity concepts. They are as follows.

- Basic capacity concept
- Indirect comparisons concept for capacity
- Comparisons concept for capacity by unit
- Measure conversion

Each core concepts consists of some concepts. These concepts are the elements of POT analysis and thy are described in Table 3.

Table 3. Contents of Capacity Concepts and Mean Score

No	Contents	Number of Total Items	Mean
	Basic Capacity Concept		
1	Definition of container	3	2.15
2	Difference between capacity and quart	2	.53
3	Direct comparisons concept for capacity	3	1.71
	Indirect Comparisons Concept for Capacity		
4	Basic indirect comparisons concept for capacity	4	3.85
5	Capacity conservation (comparative concept whereby deformities)	4	2.63
6	Capacity conservation (comparative concept whereby partitions)	2	1.43
	Comparisons Concept for Capacity by Unit		
7	General unit concept	6	4.95
8	Recognize the liquid quantity of measuring cup	5	4.17
9	Concept of estimation	2	1.13
	Measure Conversion		
10	Aggregation of capacity unit	4	3.53
11	Transformation of capacity unit	4	2.86

The items in test are dichotomous. However, each concepts are included within several items so that scoring of these concepts is polytomous. For example, concept 1 in Table 3 is included in three items. Therefore, the scoring of concept 1 is 0,1,2,3 and it has four categories.

Suppose N ($n = 1,2,\cdots,N$) examinee participate in the test and M ($m = 1,2,\cdots,M$) items which measure K ($k = 1,2,\cdots,K$) concepts. Let $X = (x_{nm})_{N\times M}$ be the response matrix in the test. $x_{nm} = 1$ means task-taker n responds item m correctly; $x_{nm} = 0$ means task-taker n responds item m incorrectly. Let $A = (a_{mk})_{M\times K}$ be the item-concept matrix. $a_{mk} = 1$ means item m measure concept k ; $a_{mk} = 0$ means item m does not measure concept k. With these matrix expressions, the scoring task-taker n on concept k is s_{nk} and $s_{nk} = \sum_{m=1}^{M} (x_{nm})(a_{mk})$. The matrix $S = (s_{nk})_{N\times K}$ is defined as follows.

$$S = X \cdot A = (x_{nm})_{N\times M} (a_{mk})_{M\times K} = (s_{nk})_{N\times K} \quad (4)$$

The researchers deal with the empirical data according to the formula above. In the POT analysis, $\varepsilon = 0.2$ is chosen so that POT will plot the graphs of concept hierarchies. As to the graphs, the relationship and hierarchies between concepts will be discussed.

5 Results

As to the discussions of results, there are two subsections. One is the basic numerical explanations of POT analysis and the other is the discussions of POT graphs.

5.1 Analysis of Ordering for All Capacity Concepts

Based on the POT calculations on $D = n'/n$ in formula (3), the percentages of disconfirmatory patterns between concepts are depicted in Table 4.

Table 4. Percentages of Disconfirmatory Pattern between Concepts

Concepts	Concepts 1	2	3	4	5	6	7	8	9	10	11
1	---	.09	.10	.69	.43	.43	.52	.58	.31	.47	.50
2	.87	---	.80	.89	.74	.69	.86	.88	.56	.78	.75
3	.46	.18	---	.98	.66	.64	.79	.84	.48	.68	.71
4	.04	.01	.02	---	.02	.04	.06	.08	.02	.04	.03
5	.47	.09	.33	.67	---	.41	.60	.64	.32	.49	.43
6	.40	.06	.35	.46	.25	---	.41	.45	.17	.34	.30
7	.24	.06	.08	.53	.20	.27	---	.36	.15	.25	.26
8	.27	.06	.16	.50	.18	.26	.39	---	.17	.23	.27
9	.57	.13	.51	.61	.42	.38	.57	.61	---	.48	.44
10	.39	.12	.31	.54	.33	.37	.49	.43	.26	---	.32
11	.34	.10	.28	.47	.24	.31	.42	.44	.19	.29	---

The tolerance level $\varepsilon = 0.2$ is chosen in this study. With this tolerance level value, the prerequisite relationship is clearly decided. It is shown in Table 5. The prerequisite relationship exists between two concepts if the subordinate value is 1.

For example, concept 1 is the prerequisite of concept 3 because the subordinate value is 1. Therefore, $1 \to 3$ will appear in the POT graph. On the contrary, concept 3 is not the prerequisite of concept 1 because the subordinate value is 0.

Table 5. Prerequisite Relationship between Concepts

Concepts	Concepts										
	1	2	3	4	5	6	7	8	9	10	11
1	---	1	1	0	0	0	0	0	0	0	0
2	0	---	0	0	0	0	0	0	0	0	0
3	0	1	---	0	0	0	0	0	0	0	0
4	1	1	1	---	1	1	1	1	1	1	1
5	0	1	0	0	---	0	0	0	0	0	0
6	0	1	0	0	0	---	0	0	1	0	0
7	0	1	1	0	0	0	---	0	1	0	0
8	0	1	1	0	1	0	0	---	1	0	0
9	0	1	0	0	0	0	0	0	---	0	0
10	0	1	0	0	0	0	0	0	0	---	0
11	0	1	0	0	0	0	0	0	1	0	---

The ordering and relationship of all capacity concepts was presented in Figure 1. Furthermore, the graph also display the mean score of concepts. It shows that the capacity concept is divided into four levels. Concept 2 is located in the highest level; concept 4 is located in the lowest level. Thus, concept 2 is the most difficult concept for students. Moreover, concept 4 is the simplest one.

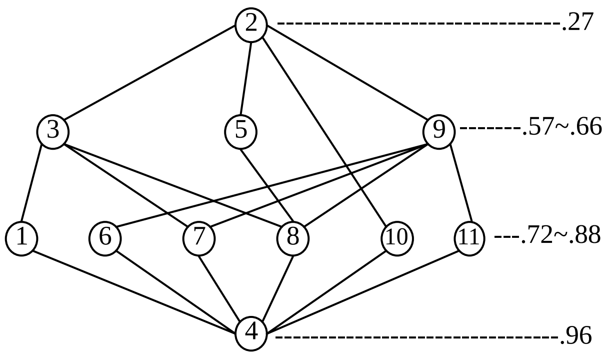

Fig. 1. The POT Graph of All Capacity Concepts

5.2 Descriptions of Ordering within the Same Core Capacity

As described in Table 3, there are four core capacity concepts and each consists of several concepts. The relationship of concepts within the same core capacity concept will be described respectively. They are discussed as follows.

5.2.1 Basic Capacity Concept

As shown in Table 3, there are three concepts within basic capacity concept. In Fig. 2, it shows that these three concepts are linear relationship. Concept 1 is a prerequisite of concept 3; concept 3 is a prerequisite of concept 2. Thus, it means that students easily own proficiency on concept 1 and concept 2 is relatively hard to master.

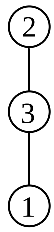

Fig. 2. The POT Graph of Basic Capacity Concept

5.2.2 Indirect Comparisons Concept for Capacity

As shown in Table 3 and Fig. 3, there are three concepts within indirect comparisons concept for capacity. Concept 4 is a prerequisite and foundation of concept 5 and concept 6. It shows that the students accomplish the proficiency of concept 4 more easily than to accomplish concept 5 and concept 6 relatively.

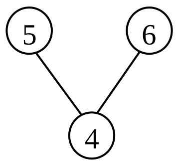

Fig. 3. The POT Graph of Indirect Comparisons Concept for Capacity

5.2.3 Comparisons Concept for Capacity by Unit

As shown in Table 3 and Fig. 4, three concepts comprise the comparisons concept for capacity by unit. Concept 7 and concept 8 are the prerequisites and basis of concept 9. This shows that the students accomplish proficiency of concept 9 more hardly than to accomplish concept 7 and concept 8 relatively.

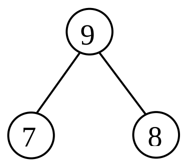

Fig. 4. The POT Graph of Comparisons Concept for Capacity by Unit

5.2.4 Measure Conversion

As shown in Table 3 and Fig. 5, there are two concepts within measure conversion. However, these two concepts, which are concept 10 and concept 11, are located in the same level and it displays independent relationship. This means that there is no precondition between these two concepts.

Fig. 5. The POT Graph of Measure Conversion

6 Conclusions

The researchers apply the extended model of ordering theory, which is POT, in the investigation of capacity concepts of pupils. The hierarchical and relational structures display the knowledge and cognition information of learning results. Based on the findings and process of investigation, some viewpoints and issues are discussed as follows.

(1) The POT graph of all capacity concepts and concepts within the same core capacity show the hierarchical structures of knowledge. The POT graphs mean the possible misconceptions of capacity knowledge. These information provided reference for remedial teaching in the classroom. Besides, how to conduct students in the learning process is also important.

(2) As to the methodology of OT, the transitivity property is an important issue. Therefore, advanced investigation on features of transitivity is needed.

(3) Except for OT, fuzzy ordering has also concerns on the hierarchy and relations among elements [13][14]. Hence, the integration and application of fuzzy ordering in the investigation of knowledge structures should be a potential study.

(4) As to viewpoints of mathematics education on the capacity concepts, they include many operation processing. Consequently, another approach of measurement and methodology, not only paper-pencil test and OT, should be quite important.

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