Robust Kalman filter of discrete-time Markovian jump system with parameter and noise uncertainty

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Abstract: Robust Kalman filtering problems for discrete-time Markovian jump systems with parameter and noise uncertainty were investigated. Because of the existence of stochastic Markovian switching, the covariance matrices of system state noise and observation noise are time-varying or unmeasurable instead of stationary, meanwhile the system suffers from structure parameter uncertainty as well. By view of robust estimation, maximum admissible upper bound of the disturbance to noise covariance matrix was given based on the estimation performance, and an optimal state estimator was therefore adopted under the worst situation. Not only can this method minimize the worst performance function of uncertainty, but also the estimation error performance can be guaranteed to be within the given precision. A numerical example shows the validity of the method.

Key–Words: Markovian jump systems;robust Kalman filter; uncertainty;

1 Introduction

Optimal filtering problem has been a hot topic in past decades, among which Kalman filtering is one of the most popular estimation approaches and considerable effort has been devoted to its theory and applications. The applications of Kalman filtering theory may be found in a large spectrum of different fields ranging from various engineering problems to biology, geoscience, economics, and management etc[1].

On the other hard, Markovian jump systems, which are convenient tools for representing many real-world systems[2]have aroused much attention in recent years. And fruitful achievements have been made in the last three decades on stability analysis[3, 4], filtering[5, 6]and control design[7, 8]. In the efforts towards filtering, Boukas[9]and Mahmoud [10]gave Kalman filtering equations for continuous-time and discrete-time Markovian jump linear systems with structure uncertainty respectively. However, in above referred contributions, all the research work was carried out based on one assumption: both the state equation and output measurement are subjected to STATIONARY Gaussian noises so that an optimal filtering gain is obtained based on the stationary noise covariance matrix. But this is not the case for Markovian jump systems. In practical environment, because of the stochastic switching in Markovian jump systems,which is usually accompanied by sudden change of working environment, the statistical characteristics(covariance matrix) of noise may be time-varying instead of stationary, and in some cases it is impossible to get the exact value of noise covariance matrix,in such cases the noise is so-called ”uncertain”. Thus the pre-designed filter may fail resulting from the change of noise covariance matrix and the controller using the estimation of system state will be incorrect. Thus the noise uncertainty will in worst case lead to system instability and a practical problem occurs: Could the pre-designed Kalman filter still efficient with the presentence of uncertain noise and parameter? How to achieve this?

In this paper, robust Kalman filtering for discrete-time Markovian jump systems under uncertain noise and parameter is considered. Firstly we give some assumptions to obtain estimation performance. Secondly we seek the maximum admissible upper bound of non-structure disturbance to noise such that the deviation of estimation performance can be within a prescribed precision. Then we discuss about the sub-optimal analytical solution by using $Lagrange$ method. Finally we prove the establishment of saddle inequality and show that our filter design is a minimax robust filter. At the end of the paper, an illustrative example is used to show the validity of our method.

2 Problem Description

Throughout the paper,unless otherwise specified,we denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e. it is right continuous and $\mathcal{F}_0$ contains all p-null sets). Let $|x|$ stand for the usual Euclidean norm for a vector x , and $|X|$ denote the Frobenius norm of a matrix X defined by $|X| = \lambda_{max}^{\frac{1}{2}}(XX^T)$, where $Tr(\cdot)$ denotes the matrix trace and the superscript T denotes transpose.Matrix $X > 0(\geq 0)$ denotes X is positive definite(semi-positive definite). Let $\{r_k, k \geq 0\}$ is a discrete Markov chain on the probability space taking values in finite state space $S = \{1, 2, \ldots, N\}$ with $P = [p_{ij}]$ the chain generator,an $N \times N$ matrix.The entries $p_{ij}, i, j \in S$ are interpreted as transition rates such that

$$p_{ij} = Pr(r_{k+1} = j | r_k = i)$$

Here $p_{ij} \geq 0$ is the transition probability from i to j . Notice that the total probability axiom imposes

$$\sum_{j=1}^{N} p_{ij} = 1, p_{ij} \geq 0 \quad \forall i \in S$$

Consider the following discrete-time Markovian jump system with uncertain noise and parameter:

$$\begin{aligned} x_{k+1} &= [A(r_k) + \Delta A(r_k)]x_k + \omega^0 \\ y_k &= [C(r_k) + \Delta C(r_k)]x_k + \upsilon^0 \end{aligned} \qquad (1)$$

where $x_k \in \mathbf{R}^n$ is state vector, $y_k \in \mathbf{R}^m$ is measurement output. $A(\cdot) \in \mathbf{R}^{n\times n}$, $C(\cdot) \in \mathbf{R}^{n\times m}$ are known matrices. ω^0, υ^0 are $n-$dimensional and $m-$dimensional white noise and satisfy the following assumption:

`Assumption 1.` For any given time $s, \tau \geq 0$, there is

(1) $E[\omega_\tau^0] = 0 \qquad E[\upsilon_\tau^0] = 0$

(2) $Cov[\omega_s^0, \omega_\tau^0] = W^0\delta_{s,\tau} = (W + \Delta W)\delta_{s,\tau}$, $W \geq 0, \Delta W \geq 0$

(3) $Cov[\upsilon_s^0, \upsilon_\tau^0] = V^0\delta_{s,\tau} = (V + \Delta V)\delta_{s,\tau}$, $V > 0, \Delta V \geq 0$

(4) $E[\begin{pmatrix} w_s^0 \\ \upsilon_s^0 \end{pmatrix} \cdot (w_\tau^{0T} \quad \upsilon_\tau^{0T})] = \begin{bmatrix} W^0\delta_{\tau,s} & 0 \\ 0 & V^0\delta_{s,\tau} \end{bmatrix}$

In Assumption 1, $W^0 \in \mathbf{R}^{n\times n}, V^0 \in \mathbf{R}^{m\times m}$consist of two parts,where W, V denote the stationary noise covariance matrix and the values are exactly known. $\Delta W, \Delta V$ denote the uncertainty caused by disturbance or time-varying,they are unknown but bounded. $\delta(\cdot,\cdot)$ is a $Dirac$ function taking values in $\{0, 1\}$.

For the deduction of Kalman filter,we introduce the following assumption([10]):

`Assumption 2.` For any fixed system mode $r_k = i \in S$, parameter(structure)uncertainty $\Delta A(i), \Delta C(i)$ satisfy

$$\begin{aligned} \Delta A(i) &= H_1(i)F(i)E(i) \\ \Delta C(i) &= H_2(i)F(i)E(i) \end{aligned} \qquad (2)$$

where $H_1(i), H_2(i), E(i), i \in S$ are known constant matrix and $F(i), i \in S$ is unknown matrix satisfying $F^T(i)F(i) \leq I$. For simplification, we denote $A(r_k = i)$, $C(r_k = i)$, $H_1(r_k = i)$, $H_2(r_k = i)$, $E(r_k = i)$, $\Delta A(r_k = i)$, $\Delta C(r_k = i)$ by A_i,C_i, H_{1i}, H_{2i}, E_i, ΔA_i, ΔC_i.

Theorem 1 *Consider stochastically stable Markovian jump system (1)and assume the noise is stationary, which means $\Delta W = \Delta V = 0$, we have the following extended Kalman filter([10]):*

$$\dot{\hat{x}} = \hat{A}_i\hat{x} + K_i[y - \hat{C}_i\hat{x}] \qquad (3)$$

where matrix Q_i, K_i are given by the following coupled Riccati equations:

$$\begin{aligned} \hat{A}_i &= A_i + (\frac{1}{\epsilon_i}H_{1i}H_{1i}^T + W)\Psi_i^{-1} \\ \hat{C}_i &= C_i + \frac{1}{\epsilon_i}H_{2i}H_{1i}\Psi_i^{-1} \\ \Psi_i &= A_i(\sum_{j=1}^{N} p_{ij}\Psi_j)A_i^T + \epsilon_i\Psi_iE_i^TE_i\Psi_i \\ &\quad + \frac{1}{\epsilon_i}H_{1i}H_{1i}^T + W \\ K_i &= (\hat{A}_i\bar{Q}_i\hat{C}_i^T + \frac{1}{\epsilon_i}H_{1i}H_{2i}^T)(\frac{1}{\epsilon_i}H_{2i}H_{2i}^T + V)^{-1} \\ Q_i &= (\hat{A}_i - K_i\hat{C}_i)\bar{Q}_i(\hat{A}_i - K_i\hat{C}_i)^T \\ &\quad + K_iVK_i^T + W \\ \bar{Q}_i &= \sum_{j=1}^{N} p_{ij}Q_j \end{aligned} \qquad (4)$$

Here parameter ϵ_i is chosen such that $tr(Q_i)$ reaches the minimum. With the above standard Kalman filter gain (3) adopted, the state estimation error satisfies:

$$E\{(x - \hat{x})(x - \hat{x}^T)\} \leq \max_{j\in S} tr(Q_j) \qquad (5)$$

Define the estimation error performance as

$$J(K_1, K_2, \ldots, K_N, W, V) = \max_{j\in S} tr(Q_j) \qquad (6)$$

According to Theorem 1 and quality of Kalman filtering, if the noise is stationary ($\Delta W = \Delta V = 0$),the estimation error performance could achieve the minimum value by adopting standard Kalman filtering (3).

However in practical world, the standard Kalman filter may fail with uncertain noise :$\Delta W \neq 0, \Delta V \neq 0$, thus the new covariance matrix of noise is W^0, V^0. If we still adopt the former pre-designed Kalman filter gain K_i,the new state estimation error should be Q_i^0, which satisfies:

$$Q_i^0 = (\hat{A}_i - K_i\hat{C}_i)\bar{Q}_i^0(\hat{A}_i - K_i\hat{C}_i)^T + K_iV^0K_i^T + W^0 \tag{7}$$

Therefore the new estimation performance is

$$J(K_1, K_2, \ldots, K_N, W^0, V^0) = \max_{j\in S} tr(Q_j^0) \tag{8}$$

According to (6) and (8), the deviation of estimation performance yielded by noise uncertainty $(\Delta W, \Delta V)$ can be written as:

$$\begin{aligned} & \Delta J(K_1, K_2, \ldots, K_N, \Delta W, \Delta V) \\ = & J(K_1, \ldots, K_N, W^0, V^0) - J(K_1, \ldots, K_N, W, V) \\ = & \max_{j\in S} tr(Q_j^0) - \max_{j\in S} tr(Q_j) \leq r \end{aligned} \tag{9}$$

Thus our purpose is: if we want the pre-designed Kalman filter still efficient under uncertain noise $(\Delta W, \Delta V)$, we should limit noise uncertainty to a certain bound. As long as the $(\Delta W, \Delta V)$ is within this bound, the admissible deviation of estimation performance $\Delta J(K_1, K_2, \ldots, K_N, \Delta W, \Delta V) \leq r$ where $r > 0$ is a design parameter according to practical requirement. In the following work, we set out to find corresponding equation between $(\Delta W, \Delta V)$ and r.

3 Upper bound of nonstructural disturbance

3.1 Expression of upper bound

According to (4), (7), we have

$$\Delta Q_i^0 = (\hat{A}_i - K_i\hat{C}_i)\Delta\bar{Q}_i^0(\hat{A}_i - K_i\hat{C}_i)^T + K_i\Delta V K_i^T + \Delta W \tag{10}$$

where $\Delta Q_i = Q_i^0 - Q_i$, from (10), it is easy to see that $tr(\Delta Q_i)$ is a linear mapping of $(\Delta W, \Delta V)$.
Define a compact convex set as $\Xi = \{(\Delta W, \Delta V) : 0 \leq \Delta W \leq \Delta W^*, 0 \leq \Delta V \leq \Delta V^*\}$, thus $\Delta J(K_1, K_2, \ldots, K_N, \Delta W, \Delta V)$ is a mapping from Ξ to $\mathbf{R}^1$, and it has following facts:
`Fact 1.` For any given $(\Delta W_j, \Delta V_j) \in \Xi, j = 1, 2$, if $\Delta W_1 \leq \Delta W_2, \Delta V_1 \leq \Delta V_2$, we have

$$\begin{aligned} & \Delta J(K_1, K_2, \ldots, K_N, \Delta W_1, \Delta V_1) \\ & \leq \Delta J(K_1, K_2, \ldots, K_N, \Delta W_2, \Delta V_2) \end{aligned}$$

`Fact 2.` Define the maximum admissible deviation of estimation performance r as

$$r = \max_{(\Delta W, \Delta V)\in\Xi} \Delta J(K_1, K_2, \ldots, K_N, \Delta W, \Delta V)$$

Thus r could be achieved only by maximum matrix pair $(\Delta W^*, \Delta V^*)$.

The purpose of following work is to construct a maximal conpact convex set Ξ^*, for any $(\Delta W, \Delta V) \in \Xi^*$,(9) is sure to establish, and a mini-max robust filtering is applied to minimize the worst performance under the noise uncertainty.

According to the finity of mode S, (9) is equivalent to

$$tr(Q_i^0) \leq r + \max_{j\in S} tr(Q_j) \tag{11}$$

Therefore for each mode $i \in S$,there is

$$\begin{aligned} tr(\Delta Q_i) &= tr(Q_i^0) - tr(Q_i) \\ &\leq r + \max_{j\in S} tr(Q_j) - tr(Q_i) \end{aligned} \tag{12}$$

Let $|\Delta W| \leq a$, $|\Delta V| \leq b$, thus we have

$$0 \leq \Delta W \leq aI_n \qquad 0 \leq \Delta V \leq bI_m$$

According to `Fact.2`, if the noise disturbance reaches the maximum aI_n, bI_m, the deviation of estimation performance will reach the maximum value r.
According to (10) and (12), we have $\forall i \in S$

$$atr(D_i) + btr(G_i) \leq r + \max_{j\in S} tr(Q_j) - tr(Q_i) \tag{13}$$

where matrix $D_i, G_i > 0, i \in S$ satisfy the following equations:

$$\begin{aligned} D_i &= (\hat{A}_i - K_i\hat{C}_i)\bar{D}_i(\hat{A}_i - K_i\hat{C}_i)^T + I_n \\ G_i &= (\hat{A}_i - K_i\hat{C}_i)\bar{G}_i(\hat{A}_i - K_i\hat{C}_i)^T + K_iK_i^T \\ \bar{D}_i &= \sum_{j=1}^{N} p_{ij}D_j \\ \bar{G}_i &= \sum_{j=1}^{N} p_{ij}G_j \end{aligned}$$

By above analysis, the seeking of admissible maximum bound of $(\Delta W, \Delta V)$ is equal to get the optimal solution of a, b such that satisfying the inequalities:

$$\begin{aligned} & max \qquad a \cdot b \\ s.t. \quad & a \cdot tr(D_i) + b \cdot tr(G_i) \leq r + \max_{j\in S} tr(Q_j) - tr(Q_i) \\ & a \geq 0 \qquad b \geq 0 \qquad i \in S \end{aligned} \tag{14}$$

Thus the the seeking of admissible maximum bound of $(\Delta W, \Delta V)$ is transformed to be a nonlinear programming problem with linear inequalities constraints. Now we discuss about the solution of this problem.

3.2 Analytical solution

Since $\Xi = \{(\Delta W, \Delta V)\}$ is a compact convex set and the inequalities in (14) compose a compact closed set on which $a \cdot b$ is defined as a continuous function.Thus there must exist the optimal solution of a, b. Decompose the original nonlinear program problem Eq. (14) into N sub-problems:

$$\begin{aligned} & max \quad a_1 \cdot b_1 \\ s.t. \quad & a_1 \cdot tr(D_1) + b_1 \cdot tr(G_1) \\ & \le r + \max_{j\in S} tr(Q_j) - tr(Q_1) \\ & max \quad a_2 \cdot b_2 \\ s.t. \quad & a_2 \cdot tr(D_2) + b_2 \cdot tr(G_2) \\ & \le r + \max_{j\in S} tr(Q_j) - tr(Q_2) \\ & \vdots \\ & max \quad a_N \cdot b_N \\ s.t. \quad & a_N \cdot tr(D_N) + b_N \cdot tr(G_N) \\ & \le r + \max_{j\in S} tr(Q_j) - tr(Q_N) \end{aligned}$$

By using $Lagrange$ method, we have the optimal analytical solution for each sub-problem as:

$$\begin{aligned} a_i^* &= \frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(D_i)} \\ b_i^* &= \frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(G_i)} \end{aligned} \tag{15}$$

Thus the analytical solution for the original nonlinear program problem Eq.(14) is taken as

$$\begin{aligned} a^* = \min_{i\in S} a_i^* = \min_{i\in S}\{\frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(D_i)}\} \\ b^* = \min_{i\in S} b_i^* = \min_{i\in S}\{\frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(G_i)}\} \end{aligned} \tag{16}$$

`Remark`: The analytical solution of the nonlinear programming problem is given by above analysis, however, it is only an optimal solution for each sub-problem. This analytical solution in Eq.(16) is local optimal but global sub-optimal. For the global optimal solution, we could only get the numerical solution by using "fmincon" function in Matlab software.The optimal analytical solution of such nonlinear programming problem is still an open problem in mathematics for further exploration.

Theorem 2 *Consider Markovian jump system (1),if we adopt state estimator (3) and Kalman filter gain (4), there exist a maximum admissible compact set Ξ. When the disturbance of noise covariance matrix $(\Delta W, \Delta V) \in \Xi$, the deviation of system state estimation performance is within a given bound r.*

4 Mini-max robust filter

Let $K_1^*, K_2^*, \ldots, K_N^*$ denotes the standard extended Kalman filtering gain corresponding to the maximum admissible noise disturbance $(\Delta W^*, \Delta V^*)$, according to the quality of Kalman filtering, we have

$$\begin{aligned} & \Delta J(K_1^*, K_2^*, \ldots, K_N^*, \Delta W^*, \Delta V^*) \\ \le \quad & \Delta J(K_1, K_2, \ldots, K_N, \Delta W^*, \Delta V^*) \end{aligned}$$

On the other hand, according to `Fact 1`, we have

$$\begin{aligned} & \Delta J(K_1^*, K_2^*, \ldots, K_N^*, \Delta W, \Delta V) \\ \le \quad & \Delta J(K_1^*, K_2^*, \ldots, K_N^*, \Delta W^*, \Delta V^*) \end{aligned}$$

Thus we have the following saddle point inequality:

$$\begin{aligned} & \Delta J(K_1^*, K_2^*, \ldots, K_N^*, \Delta W, \Delta V) \\ & \le \Delta J(K_1^*, K_2^*, \ldots, K_N^*, \Delta W^*, \Delta V^*) \\ & \le \Delta J(K_1, K_2, \ldots, K_N, \Delta W^*, \Delta V^*) \end{aligned}$$

By Game theory, the optimal estimator under the worst situation is the mini-max estimator:

$$\begin{aligned} & \min_{K_i} \max_{(\Delta W, \Delta V)\in\Xi} \Delta J(K_1, K_2, \ldots, K_N, \Delta W, \Delta V) \\ = & \max_{(\Delta W, \Delta V)\in\Xi} \min_{K_i} \Delta J(K_1, K_2, \ldots, K_N, \Delta W, \Delta V) \end{aligned}$$

5 Simulation

Consider the following two-mode discrete-time Markovian jump system:
Let the system mode $r_k = 1$ be given by

$$A_1 = \begin{bmatrix} 0.7 & 0 \\ 0 & 1.1 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 1.5 & 0 \end{bmatrix}$$

$$H_1(1) = [0.1 \quad 0.1]^T \quad H_2(1) = 0.1 \quad E_1 = [0.3 \quad 0.2]$$

Let the system mode 2 be given by

$$A_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 1 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$H_1(2) = [0.15 \quad 0.15]^T \quad H_2(2) = 0.1 \quad E_2 = [0.2 \quad 0.2]$$

Stationary noise covariance matrix and mode transition matrix is

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, V = 1, P = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.3 \end{bmatrix}$$

The admissible bound of performance deviation is $r = 0.3$

1)Solve the equation (4),get Q_1, Q_2 and K_1, K_2:

$$Q_1 = \begin{bmatrix} 2.4188 & 1.2012 \\ 1.2012 & 2.7563 \end{bmatrix}, K_1 = \begin{bmatrix} 0.6590 \\ 1.0087 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 2.6753 & 1.2531 \\ 1.2531 & 2.8966 \end{bmatrix}, K_2 = \begin{bmatrix} 0.6754 \\ 0.5096 \end{bmatrix}$$

2)Substitute the result to (14), by using $Lagrange$ method, the upper bound of noise uncertainty are given as: $a^* = 0.1493, b^* = 0.1677$

3)Let the noise covariance matrix correspond to the maximum perturbation:

$W^* = W + \Delta W = W + a^* \cdot I_2,$

$V^* = V + \Delta V = V + b^* \cdot I_1$

4)Repeat step 2), and we have the correspondent $Q_1^*, Q_2^*, K_1^*, K_2^*$ for new noise covariance matrix (W^*, V^*):

$$Q_1^* = \begin{bmatrix} 2.4351 & 1.2235 \\ 1.2235 & 2.7898 \end{bmatrix}, K_1^* = \begin{bmatrix} 0.6903 \\ 1.1458 \end{bmatrix}$$

$$Q_2^* = \begin{bmatrix} 2.7858 & 1.2658 \\ 1.2658 & 3.0177 \end{bmatrix}, K_2^* = \begin{bmatrix} 0.7012 \\ 0.5241 \end{bmatrix}$$

5)With the robust Kalman filtering applying:

$\Delta J(K_1^*, K_2^*, \Delta W^*, \Delta V^*) = 0.2316 < 0.3$

6 Conclusion

In this paper, the robust Kalman filter for discrete-time Markovian jump system with uncertain noise and parameter is considered. A new method is given to obtain the maximum admissible bound of the disturbance to noise so that the deviation of estimation performance is guaranteed to be within a given precision. The seeking of bound to noise uncertainty could be transformed to a nonlinear programming problem, and the analytical solution of this problem is also discussed in this paper, which is a sub-optimal and conservative solution via $Lagrange$ method. The simulation show the validity of this design method.

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