Some Properties of Fractional Bessel Processes Driven by Fractional Brownian Motion

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Abstract: Fractional Bessel processes are defined and considering the processes associated with fractional Bessel processes

$$X_H(t) = \int_0^t sign(B_H(s))dB_H(t), \frac{1}{2} < H < 1$$

$$Y_H(t) = \sum_{j=1}^{d} \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s)$$

where $B_H = (B_H(1), B_H(2), ..., B_H(d))$ *is a d-dimensional* ($d \geq 2$) *fractional Brownian motion with Hurst parameter 0<H<1 and* $R_H = \sqrt{B_H(1)^2 + B_H(2)^2 + ... + B_H(d)^2}$ *is fractional Bessel Process driven by fractional Brownian Motion, some of their properties are given.*

Key-Words: fractional Brownian motion, fractional Bessel processes, local time, Tanaka formula

1 Introduction

First we will give the definition of Bessel processes and Fractional Brownian Motion.

For every $\delta \geqslant 0$ and $x \geqslant 0$, the solution to the equation

$$X_t = x + \delta t + 2\int_0^t \sqrt{|X_s|} dW_s$$

is unique and strong. In the case δ =0, *x* = 0, the solution X_t is identically zero and applying the comparison theorem (see Revuz–Yor [11] Theorem IX.(3.7)) we conclude $X_t \geq 0$ for all $\delta \geq 0$.

Definition 1 ($BESQ^\delta$) For every $\delta \geq 0$ and x $\geqslant 0$ the unique strong solution to the equation

$$X_t = x + \delta t + 2\int_0^t \sqrt{|X_s|} dW_s$$

is called the square of a δ-dimensional Bessel process started at *x* and is denoted by $BESQ^\delta$.

Remark: the law of $BESQ^\delta(x)$ on $C(R_+, R)$ by Q_x^δ. We call the number δ the *dimension* of $BESQ$.

This notation arises from the fact that a $BESQ^\delta$ process X_t can be represented by the square of the Euclidean norm of δ-dimensional Brownian motion B_t: $X_t = ||B_t|^2|$. The number $\nu \equiv \delta/2 - 1$ is called the *index of the process* $BESQ^\delta$.

Definition 2 (BES^δ) The square root of $BESQ^\delta(a^2)$, $\delta \geq 0$, $a \geq 0$ is called the Bessel process of dimension δ *started at* a *and is denoted by* $BES^\delta(a)$.

Remark: the law of $BES^\delta(a)$ by P_a^δ

In the case $\delta \geq 2$, $BES^\delta(a)$, $a > 0$, will never reach 0. For $\delta > 1$ a $BES^\delta(a)$ process Z_t satisfies $E[\int_0^t (ds/Z_s)] < \infty$ and is the solution to the equation

$$Z_t = a + \frac{\delta - 1}{2}\int_0^t \frac{ds}{Z_s} + W_t$$

For $\delta \leq 1$ the situation is less simple. For $\delta = 1$ we have with *Itô* Tanaka's formula

$$Z_t = |W_t| = \widetilde{W}_t + L_t$$

where $\widetilde{W}_t \equiv \int_0^t sign(W_s)dW_s$ is a standard Brownian motion, and *Lt* is the local time of Brownian motion.

Refer to Revuz–Yor [11] and Pitman–Yor [9, 10] for the more study of Bessel processes.

Definition 3 (*fBm*) Let $H \in (0,1)$ be a constant. *The (1-parameter) fractional Brownian motion* (*fBm*) with Hurst parameter H is the Gaussian process $B_H(t) = B_H(t,\omega), t \in R, \omega \in \Omega$, satisfying $B_H(0) = E[B_H(t)] = 0$, for all $t \in R$.
and

$$E[B_H(s)B_H(t)] = \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}; s,t \in R$$

Where E denotes the expectation with respect to the probability law P for $\{B_H(t,\omega); t \in R, \omega \in \Omega\}$, where (Ω, F) is a measurable space.

If H = 1/2 then $B_H(t)$ coincides with the classical Brownian motion, denoted by $B(t)$.

If H > 1/2 then $B_H(t)$ is persistent, in the sense that $\rho_n = E[B_H(1)\cdot B_H(n+1) - B_H(n)] > 0$ for all n = 1, 2, . . .

and $\sum_{n=1}^{\infty} \rho_n = \infty$

If H < 1/2 then $B_H(t)$ is anti-persistent, in the sense that

$\rho_n < 0$ for all n = 1, 2, . . .

in this case $\sum_{n=1}^{\infty} \rho_n < \infty$ (Shiryaev [5], p. 233)

Another important property of *fBm* is *self-similarity*: For any $H \in (0,1)$ and $\alpha > 0$ the law of $\{B_H(\alpha t)\}_{t \in R}$ is the same as the law of $\{\alpha^H B_H(t)\}_{t \in R}$.

Definition 4 Denote the *fractional Bessel process* by

$$R_H = \sqrt{B_H(1)^2 + B_H(2)^2 + \ldots + B_H(d)^2}$$

where $B_H = (B_H(1), B_H(2), \ldots, B_H(d))$ be a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

We hope to obtain a stochastic calculus for *fBm* and to use its properties into application.

However, if $H \neq 1/2$ then $B_H(t)$ is not a semimartingale, so we cannot use the general theory of stochastic calculus for semimartingales on $B_H(t)$. For example, as $H \neq 1/2$ the fractional Brownian motion $B_H(t)$ has not $L\hat{e}vy$ type characteristic, i.e., the process (see Hu [7])

$$X_H = \int_0^t sign(B_H(s))dB_H(s), \frac{1}{2} < H < 1$$

(1.1)

is not a *fBm*. Furthermore, the process

$$Y_H(t) = \sum_{j=1}^{d} \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s) \tag{1.2}$$

is the fractional Bessel process. Thus, it is interesting to investigate the properties of these processes. Hu and Nualart obtained some properties of these processes in [7].

The purpose of this paper is to prove the local times of these processes based on $B_H(t)$ exist, 1/2 < H < 1. Moreover, we give a Tanaka formula of the process X_H given by (1.1) and (1.2).

2 Fractional *Itô* type stochastic integral

For 1/2 < H < 1, an alternative integration theory based on the Wick product ◇ was introduced by [3], as follows:

$$\int_0^t u(s)dB_H(s) := \lim_{|\pi_n|\to 0} \sum_k u(t_k) \Diamond (B_H(t_{k+1}) - B_H(t_k))$$

Where $\pi_n : 0 \le t_0 \le t_1 \le \ldots \le t_n = t$ is an arbitrary partition of [0, t], $\pi_n := \max_k \{t_{k+1} - t_k\}$ and $\lim_{|\pi_n|\to 0}$ means the limit in $L^2(\mu)$. The definition of the integrals has been extended by [8] (see also [1]) to all 0 < H < 1 as follows:

$$\int_0^t u(s)dB_H(s) := \int_0^t u(s) \Diamond W_{(H)}(s)ds$$

where $W_{(H)}(t) = \dfrac{dB_H(t)}{dt} \in (S)^*$ with $(S)^*$ the Hida space of stochastic distributions if $u : R_+ \to (S)^*$ satisfies that $u(t) \Diamond W^H(t)$ is dt-integrable in $(S)^*$. These fractional *Itô* integrals have many properties of the classical *Itô* integral.

Recall that the Malliavin Φ -derivative of the function $U: \Omega \to R$ defined in [3] as follows:

$$D_s^\phi U = \int_0^\infty \phi(r,s) D_r U dr$$

where $D_r U$ is the fractional Malliavin derivative at r. Define the space $L_\phi^{1,2}$ to be the

set of measurable processes u such that $D_s^\phi u(s)$ exists for a.a. $s \ge 0$ and

$$E[(\int_0^\infty D_s^\phi u(s)ds)^2 + \int_0^\infty \int_0^\infty u(s_1)u(s_2)\phi(s_1,s_2)ds_1 ds_2] < \infty$$

Then the integral $\int_0^\infty u(s)dB_H(s)$ can be well defined as an element of $L^2(\mu)$

Theorem 2.1 ([3]). Let $\{u(t), t \geq 0\}$ be a stochastic process in $L_\phi^{1,2}$. Then for the process

$$\eta(t) = \int_0^\infty u(s)dB_H(s), t \geq 0$$

we have

$$D_s^\phi \eta(t) = \int_0^t u(r)dB_H(r) + \int_0^t u_r \phi(s,r)dr$$

In particular, if *u* is deterministic, then

$$D_s^\phi \eta(t) = \int_0^t u(r)\phi(s,r)dr$$

3 Local Time and Tanaka Formula

Refer to [9], the weighted local time $L(B_H)$ of fractional Brownian motion are established:

$$L(B_H) = 2H\int_0^t \delta(B_H(s) - x)s^{2H-1}ds$$

The Tanaka formula is given as:

$$(B_H(t) - x)^+ = x^+ + \int_0^t 1_{\{B_H(s) > x\}} dB_H(s) + \frac{1}{2}L_t^x(B_H)$$

$$| B_H(t) - x | = | x | + \int_0^t sign(B_H(s))dB_H(s) + L_t^x(B_H)$$

In this section we show that the local times of the process

$$X_H = \int_0^t sign(B_H(s))dB_H(s), t \geq 0$$

$$Y_t^H = \sum_{j=1}^d \int_0^t \frac{B_s^H}{R_s^H} dB_s^H(j)$$

exist and obtain their Tanaka formula.

Lemma 3.1 (Hu [7])

$$E[sign(B_H(s))sign(B_H(u))] = \sum_{k=0}^\infty \frac{4(2k)!(s^{2H} + u^{2H} - |s-u|^{2k+1})}{(2k+1)^2 2\pi(k!2^k)^2(su)^{(2k+1)}}, t \geq 0.$$

We can get the proof of this Lemma in [7]. By using this Lemma its easy to show the following result holds

Lemma 3.2 Let $1/2 < H < 1$, then $sign(B_H(t))D_H X_H(t) \geq 0, a.s$ for all $t \geq 0$.

Theorem 3.1. Let $\Phi : R^+ \to R$ be a convex function having polynomial growth and let the process X_H be defined by

$X_H(t) = \int_0^t sign(B_H(s))dB_H(s), t \geq 0$ Then there exists a continuous increasing process A^Φ such that

$$\Phi(X_H(t)) = \Phi(0) + \int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s) + \frac{1}{2}A_t^\Phi, t \geq 0$$

where $D^-\Phi$ denotes the left-hand derivative of Φ

Proof: If $\Phi \in C^2$, then this is the *Itô* formula and

$$A_t^\Phi = \int_0^t \Phi''(X_s)sign(B_H(s))D_H X_H(s)ds$$

and Lemma 3.1 implies that the process A^Φ is increasing.

Let now $\Phi \notin C^2$. For $\varepsilon > 0$ and $x \in R$ we set

$$p_\varepsilon(\varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{1}{2\varepsilon}x^2}$$

and

$$\Phi_\varepsilon(x) = \int_R p_\varepsilon(x-y)\Phi(y)dy, (\varepsilon > 0)$$

Then $\Phi_\varepsilon(x)$ has polynomial growth and $\Phi_\varepsilon \in C^2$. It follows that for all $\varepsilon > 0$ there exists a continuous increasing process A^Φ such that

$$\Phi_\varepsilon(X_H(t)) = \Phi_\varepsilon(0) + \int_0^t \Phi'_\varepsilon(X(s))sign(B_H(s))dB_H(s) + \frac{1}{2}A_t^{\Phi_\varepsilon}$$

and

$$A_t^{\Phi_\varepsilon} = \int_0^t \Phi_\varepsilon''(X_H(s))sign(B_H(s))D_H X_H(s)ds$$

$$= \int_R \Phi_\varepsilon''(x)(\int_0^t \delta(X_H(s) - x)(sign(B_H(s))D_H X_H(s)ds)dx$$

Noting that for all $x \in R$

$$\lim_{\varepsilon \downarrow 0} \Phi_\varepsilon(x) = \Phi(x), \quad \lim_{\varepsilon \downarrow 0} \Phi'_\varepsilon(x) = D^-\Phi(x)$$

So as $\varepsilon \to 0$.

$$\int_0^t \Phi'_\varepsilon(X_H(s))sign(B_H(s))dB_H(s) \to \int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s)$$

in probability. As a result, $A_t^{\Phi_\varepsilon}$ converges also to a process A^Φ which, as a limit of increasing processes, is itself an increasing process and

$\Phi(X_H(t)) = \Phi(0) +$

$\int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s) + \frac{1}{2}A_t^{\Phi_\varepsilon}$

The process A^Φ can now obviously be chosen to be a.s. continuous. This completes the proof.

Corollary 3.1. For any real number *x*, there exists an increasing continuous process $L^x(X_H)$ called the local time of the process X_H given by (1.1) in *x* such that,

$|X_H(t) - x|$

$= |x| + \int_0^t sign(X_H(s) - x)dX_H(s) + L_t^x(X^H)$.

Combining this corollary with [3, 9], we get the following

Corollary 3.2. Let $L(X)$ denote the local time of the process X and let

$L_t^x(B_H) = 2H\int_0^t \delta(B_H(s) - x)s^{2H-1}ds$

be the weighted local time of fractional Brownian motion B_H. Then we have

$L_t^x(X_H) - L_t^x(B_H)$

$= |X_H(t) - x| - |B_H(t) - x|$

$+ 2\int_0^t 1_{\{X_s \le x\}} sign(B_H(s) - x)dB_H(s)$

Corollary 3.3. For any real number *x* and $t \ge 0$, we have

$L_t^x(X_H)$

$= \int_0^t \delta(X_H(s) - x)sign(B_H(s))D_H(s)X_H(s)ds$.

Moreover, for any convex function having polynomial growth $\Phi : R^+ \to R$ the following Ito-Tanaka type formula holds:

$\Phi(X_H(t))$

$= \Phi(0) + \int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s)$

$+ \frac{1}{2}\int_R L_t^x(X_H)\mu_\Phi(dx)$

where $D^-\Phi$ denotes the left derivative of Φ and the signed measure μ_Φ is defined by

$\mu_\Phi([a,b]) = D^-\Phi(b) - D^-\Phi(a), a < b, a, b \in R$

Finally, by the same method on can show that the local time of the process

$Y_H(t) = \sum_{j=1}^d \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s)$

holds, where $B_H = (B_H(1), B_H(2), ..., B_H(d))$ is a $d(\ge 2)$ dimensional fractional Brownian motion with Hurst index $1/2 < H < 1$ and

$R_H = \sqrt{B_H(1)^2 + B_H(2)^2 + ... + B_H(d)^2}$ is the fractional Bessel process.

4 Conclusion

It can be seen from the above-mentioned analysis that the processes associated with fractional Bessel processes

$X_H(t) = \int_0^t sign(B_H(s))dB_H(t), \frac{1}{2} < H < 1$

$Y_H(t) = \sum_{j=1}^d \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s)$

where $B_H = (B_H(1), B_H(2), ..., B_H(d))$ converge, have the local times $L^x(X_H)$ and Ito-Tanaka type formula

$\Phi(X_H(t))$

$= \Phi(0) + \int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s)$

$+ \frac{1}{2}\int_R L_t^x(X_H)\mu_\Phi(dx)$

holds.

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