A preliminary approach to deterministic identifiability of uncontrolled linear switching systems

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Abstract: - In certain optical biosensor experiments, the dynamics may be modeled by a linear switching system. There is not an obvious test of a linear switching system for deterministic identifiability in the systems theory literature. This is an initial attempt at proposing a workable test for this property. The application of interest invites study of uncontrolled linear switching systems of two phases.

Key–Words: Deterministic identifiability, Linear switching system, Linear compartmental system

1 Introduction

Optical biosensors provide a means of studying interactions of chemical species. Experimental data may be used to estimate rate constants of these interactions. Certain widely used biosensors allow kinetic experiments of two phases and the dynamics of each is modeled by a system of linear time invariant differential equations which are compartmental in type. It appears that linear switching systems (LSS) provide an appropriate framework for modeling the output of the biosensor.

Prior to data collection and estimation of parameters, it is natural to consider if any models used are deterministically identifiable. A notion of identifiability for linear switching systems is foreign to the systems theory literature. In this preliminary study, the application of interest suggests investigation of uncontrolled LSS of two phases. It is seen that the standard type of definition of deterministic identifiability does not engender an obvious test for this property.

Out of necessity, a definition for deterministic identifiability of a LSS is proposed. The test of the system for identifiability exploits its piecewise linear time invariant nature. A test case is considered for which the constituent subsystems are not structured. Aspects of the control theory literature suggests that not all identifiability tests available for testing linear time invariant systems are suitable for unstructured systems. Ultimately an adaptation of the Laplace transform approach allows classification of the system of interest as deterministically globally identifiable.

2 Preliminaries

The following sets are used in this paper:
$\mathbb{R}_+ = \{x : x \geq 0, x \in \mathbb{R}\}$,
$\bar{\mathbb{R}}_+ = \{x : x > 0, x \in \mathbb{R}\}$,
$\mathcal{D}_a = \{s : s \in \mathbb{C}, Re(s) > a\}$.
$H_{t_1}(\cdot)$ represents the Heaviside step function with unit jump at $t = t_1$.
$\mathbf{I}_n$ represents the $n \times n$ identity matrix.

2.1 Classes of linear time invariant systems

van den Hof gives a very readable sequence of definitions which deftly divides adjective accumulation into manageable portions [10]. For this reason they are reproduced below, with some minor adjustment and some hair-splitting.

Definition 1 *A linear dynamic system (continuous time, time invariant linear system in state space form) is a dynamic system with state, input and output spaces $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and $Y = \mathbb{R}^k$ and time index set $T \subseteq \mathbb{R}$. The state, input and output functions are related by*

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x_0}, \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{k\times n}$, $\mathbf{D} \in \mathbb{R}^{k\times m}$.

For this paper it is useful to introduce the class of parametric linear dynamic systems and then impose a condition on these to define structured linear dynamic systems.

Definition 2 *A parametric linear dynamic system is a family of linear dynamic systems together with a parameter set* $\Theta \subset \mathbb{R}^p$ *for some* $p \in \mathbb{N}$ *and maps*

$$\begin{aligned} \mathbf{A} &: \Theta \to \mathbb{R}^{n\times n}, \mathbf{B} : \Theta \to \mathbb{R}^{n\times m}, \mathbf{C} : \Theta \to \mathbb{R}^{k\times n}, \\ \mathbf{D} &: \Theta \to \mathbb{R}^{k\times m}, \mathbf{x_0} : \Theta \to \mathbb{R}^n . \end{aligned}$$

The parametric linear dynamic system will be represented by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\boldsymbol{\theta})\mathbf{x}(t) + \mathbf{B}(\boldsymbol{\theta})\mathbf{u}(t), \\ \mathbf{x}(0) &= \mathbf{x_0}(\boldsymbol{\theta}), \\ \mathbf{y}(t) &= \mathbf{C}(\boldsymbol{\theta})\mathbf{x}(t) + \mathbf{D}(\boldsymbol{\theta})\mathbf{u}(t). \end{aligned} \tag{2}$$

Definition 3 *A structured linear dynamic system is a parametric linear system in which the elements of the system matrices are either fixed zeros or free parameters.*

Remark 4 *It is stated in van den Hof that "A subclass of the linear dynamic systems is formed by the positive linear systems [10]. Positive linear systems are linear dynamic systems in which the state, input and output spaces are* $X = \mathbb{R}^n_+$, $U = \mathbb{R}^m_+$ *and* $Y = \mathbb{R}^k_+$ *respectively." A complication arises as a linear system does not have to be time invariant to satisfy the above conditions.*

Rather than introduce a significant amount of systems theory to formally define positive linear systems, for this paper it is sufficient to consider positive linear time invariant systems, derived somewhat informally from the linear dynamical systems.

Definition 5 *Define a parametric positive linear time invariant system by considering a parametric linear dynamic system as in definition 2 modified such that the state, input and output spaces are* $X = \mathbb{R}^n_+$, $U = \mathbb{R}^m_+$ *and* $Y = \mathbb{R}^k_+$ *respectively.*

Definition 6 *A linear time invariant compartmental system is a parametric positive linear time invariant system after definition 5 of the form (2) for which conservation of mass holds. This corresponds with the following requirements on the matrices of the system.*

- *All elements of* $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$ *are non-negative.*
- *for* $\mathbf{A} = (a_{i,j})_{i,j=1,\ldots,n}$,

$$\begin{aligned} a_{ij} &\geq 0, & i,j \in \{1,\ldots,n\}, i \neq j, \\ a_{ii} &\leq -\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ji} & i \in \{1,\ldots,n\}. \end{aligned} \tag{3}$$

Remark 7 *A compartmental system may be structured. If the compartmental system is* <u>*closed*</u> *then it is certainly not structured as the elements of* $\mathbf{A}$ *are interrelated by* $a_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ji} \;\; i \in \{1,\ldots,n\}$.

2.2 Linear Switching System

A class of linear hybrid systems of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_{\gamma(t)}\mathbf{x}(t) + \mathbf{B}_{\gamma(t)}\mathbf{u}(t), \\ \mathbf{x}(0) &= \mathbf{x_0}, \\ \mathbf{y}(t) &= \mathbf{C}_{\gamma(t)}\mathbf{x}(t), \\ \text{where } \gamma &: T \to \Gamma : t \mapsto \gamma(t) \in \Gamma. \end{aligned} \tag{4}$$

are presented in Ezzine and Haddad [4]. Assume the state, output and input spaces and dimensions of the system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are as in definition 1.

Consider a parametric uncontrolled linear switching system of two phases $M(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$, derived from equation (4). For $i = 1, 2$, consider suitable sets Θ_i and $\boldsymbol{\theta_i} \in \Theta_i$. Denote $\boldsymbol{\theta} \in \Theta$ as the vector of parameters obtained by concatenating $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$ and removing any repeated elements. The states and outputs are related by

$$\begin{aligned} \dot{\mathbf{x}}(t,\boldsymbol{\theta}) &= \mathbf{A}_{\gamma(t)}(\boldsymbol{\theta}_{\gamma(t)})\mathbf{x}(t,\boldsymbol{\theta}), \\ \mathbf{x}(0,\boldsymbol{\theta}) &= \mathbf{x_0}(\boldsymbol{\theta_1}), \\ \mathbf{y}(t,\boldsymbol{\theta}) &= \mathbf{C}_{\gamma(t)}(\boldsymbol{\theta}_{\gamma(t)})\mathbf{x}(t,\boldsymbol{\theta}), \end{aligned} \tag{5}$$

with switching function defined by

$$\gamma(t) = \begin{cases} 1 & 0 \leq t < t_1, \\ 2 & t \geq t_1, \end{cases}$$

$t_1 > 0$ is the switching time. It is assumed that $\mathbf{x}(t_1, \boldsymbol{\theta}_1) = \mathbf{x}(t_1^-, \boldsymbol{\theta}_1)$.

2.3 Terminological issues

2.3.1 Structural properties

The adjective "structural" has multiple meanings in the systems theory literature. A structural property of a system is one which holds for almost all $\boldsymbol{\theta} \in \Theta$, that is, everywhere except for possibly sets of measure zero.

As already seen, "structural" is also used as a description for types of systems. In Hovelaque *et al.* it is stated that structured systems can be studied for structural properties [7]. Yamada and Luenberger allude to limitations of structural controllability as it assumes the system is structured [11].

2.3.2 Types of identifiability

Broadly speaking, a parametric model or system is classed as having the property of global identifiability if it is possible to uniquely determine the parameters from output obtained under a prescribed set of conditions. A model represents a process in the compartmental analysis literature. Conditions used to test a model for identifiability include: the model structure correctly represents the process and the system produces error free output for a certain period of time. A deterministic system is defined by some set of equations, it does not have uncertain structure or output subject to error. The conditions used in testing a system for identifiability do not require those used for a model. This paper will consider system identifiability.

There are a variety of definitions for "structural identifiability" of a system in the literature in which structural has additional meaning. The original treatment of structural identifiability appeared in Bellman and Åström [2]. It is implied that the essential properties of the input-output map of the system are determinable by observing output when a sufficiently rich set of inputs may be applied.

Alternatively, Godfrey's "deterministic identifiability" considers a situation where only specified inputs may be applied to the system [5] . In the case of uncontrolled systems, structural and deterministic identifiability may be effectively equivalent. The term structural is laden with connotations, not all of which are suitable for the system of interest. To avoid any possible misunderstanding, the convention of "deterministic identifiability" is imposed in this paper.

2.4 Deterministic identifiability of uncontrolled systems

Consider a system $M(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$ with state $\mathbf{x}(t,\boldsymbol{\theta}) \in \mathbb{R}^n$ and output $\mathbf{y}(t,\boldsymbol{\theta}) \in \mathbb{R}^k$. The system in state space form is described by

$$\begin{aligned}\dot{\mathbf{x}}(t,\boldsymbol{\theta}) &= \mathbf{f}(\mathbf{x}(t,\boldsymbol{\theta}),\boldsymbol{\theta}), \ \ \mathbf{x}(0,\boldsymbol{\theta}) = \mathbf{x}_0(\boldsymbol{\theta}),\\ \mathbf{y}(t,\boldsymbol{\theta}) &= \mathbf{h}(\mathbf{x}(t,\boldsymbol{\theta}),\boldsymbol{\theta}).\end{aligned}$$

The following conditions are required of M, following Denis-Vidal and Joly-Blanchard, [3]. Suppose Θ is an open subset of $\mathbb{R}^p$, $p \in \mathbb{N}$. The functions $\mathbf{f}(\cdot,\boldsymbol{\theta})$, $\mathbf{h}(\cdot,\boldsymbol{\theta})$ are real and analytic for every $\boldsymbol{\theta} \in \Theta$ on S (a connected open subset of $\mathbb{R}^n$ such that $\mathbf{x}(t,\boldsymbol{\theta}) \in S$ for every $\boldsymbol{\theta} \in \Theta$ and every $t \in [0, T_{max}]$). It is also assumed that $\mathbf{f}(\mathbf{x}_0(\boldsymbol{\theta}),\boldsymbol{\theta}) \neq \mathbf{0}$ for every $\boldsymbol{\theta} \in \Theta$.

When discussing system identifiability, it is usual to solve some equations. In this paper the definition of system identifiability is reformulated slightly to give the solution set a name but is quite similar to that in [3].

Definition 8 *Consider system $M(\boldsymbol{\theta})$. Define the set*

$$ID(M)=\left\{\boldsymbol{\theta}' : \mathbf{y}(t,\boldsymbol{\theta})=\mathbf{y}(t,\boldsymbol{\theta}'), \forall t\in[0,\tau],\ \boldsymbol{\theta}',\boldsymbol{\theta}\in\Theta\right\}. \quad (6)$$

M is deterministically globally identifiable if for almost all $\boldsymbol{\theta} \in \Theta$, $ID(M) = \{\boldsymbol{\theta}\}$.
M is deterministically locally identifiable if for almost all $\boldsymbol{\theta} \in \Theta$, the elements of $ID(M)$ are denumerable.
M is deterministically unidentifiable if for almost all $\boldsymbol{\theta} \in \Theta$, the elements of $ID(M)$ are not denumerable.

It is also possible to define identifiability by considering essential features of the output function, as seen in the approach of Jacquez and Greif [8]. System output $\mathbf{y}(\cdot,\boldsymbol{\theta})$ is expressed as a function of time and observational parameters $\boldsymbol{\phi}(\boldsymbol{\theta})$,

$$\mathbf{y}(t,\boldsymbol{\theta}) = \mathbf{g}(\boldsymbol{\phi}(\boldsymbol{\theta}),t). \quad (7)$$

By definition, the elements of $\boldsymbol{\phi}$ are uniquely determinable. Equation (7) allows an alternative definition for deterministic identifiability of a system.

Definition 9 *Suppose the output of system $M(\boldsymbol{\theta})$ features observational parameters $\boldsymbol{\phi}(\boldsymbol{\theta})$. Define the set*

$$ID(M,\boldsymbol{\phi})=ID(M)=\left\{\boldsymbol{\theta}' : \boldsymbol{\phi}(\boldsymbol{\theta})=\boldsymbol{\phi}(\boldsymbol{\theta}'),\boldsymbol{\theta}',\boldsymbol{\theta}\in\Theta\right\}. \quad (8)$$

Testing M for identifiability requires solution of algebraic equations. Once $ID(M,\boldsymbol{\phi})$ is found, M is classified as before by definition 8.

2.4.1 Modification for switching systems

Consider a LSS for which the output is expressible in terms of equation (7). Such a system may be considered as a degenerate form of LSS which has a representation as a linear time invariant system. Such systems are not the concern of this paper. Consider a linear switching system of two phases of equation (5) which is not of the degenerate type. The output of this system is expressible as

$$\mathbf{y}(t) = \begin{cases} \mathbf{g_1}(\boldsymbol{\phi}_1(\boldsymbol{\theta}_1),t) & t \in [0,t_1),\\ \mathbf{g_2}(\boldsymbol{\phi}_2(\boldsymbol{\theta}),t) & t \in [t_1,\infty).\end{cases}$$

Some care is required to ensure the conditions of section 2.4 are satisfied. The first and second constituent sub-systems are linear, real and analytic on their respective intervals $(0, t_1^-)$ and (t_1, ∞). The condition that $\dot{\mathbf{x}}(0, \boldsymbol{\theta}) \neq \mathbf{0}\ \forall \boldsymbol{\theta} \in \Theta$ applies for the first subsystem and translates to $\dot{\mathbf{x}}(t_1, \boldsymbol{\theta}) \neq \mathbf{0}\ \forall \boldsymbol{\theta} \in \Theta$ for the second.

Definition 9 requires some re-interpretation for switching systems. Suppose $\boldsymbol{\phi}$ in equation (8) is redefined for this setting as

$$\boldsymbol{\phi}(\boldsymbol{\theta}) = \left(\boldsymbol{\phi_1}(\boldsymbol{\theta_1})^\top, \boldsymbol{\phi_2}(\boldsymbol{\theta})^\top\right)^\top. \tag{9}$$

where the elements of $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are assumed to be uniquely determinable.

2.5 Survey of existing tests for identifiability

The identifiability literature does not have a test which is explicitly intended for linear switching systems. One may consider if there is any suitable test from the area of time-varying linear systems.

The linear switching system of two phases could be considered as a linear time-varying system where the elements of the system matrices are functions of the form $\omega(\cdot) = K\left(H_0(\cdot) - H_{t_1}(\cdot)\right)$, for K constant. The technique of Audoly *et al.* is suitable for certain types of time-varying parameters but does not seem immediately applicable to parameters with jumps as the method appears to require that expressions for time-varying parameters are differentiable with respect to time [1].

As the LSS is piecewise linear time invariant (LTI), it seems reasonable to turn to this field for inspiration. The nature of the constituent systems of the LSS makes it possible to express the output of the system in terms of functions which are like LTI system outputs. This property is revealed clearly by defining

$$\begin{aligned}
\mathbf{x_1} : \mathbb{R}_+ \to \mathbb{R}^n, \mathbf{x_1}(t, \boldsymbol{\theta}_1) &= e^{\mathbf{A}_1(\boldsymbol{\theta}_1)t}\mathbf{x_0}(\boldsymbol{\theta_1}),\\
\mathbf{y_1} : \mathbb{R}_+ \to \mathbb{R}^m, \mathbf{y_1}(t, \boldsymbol{\theta}_1) &= \mathbf{C_1}(\boldsymbol{\theta_1})\mathbf{x_1}(t, \boldsymbol{\theta}_1),\\
\mathbf{x_2} : \mathbb{R}_+ \to \mathbb{R}^n, \mathbf{x_2}(t, \boldsymbol{\theta}) &= e^{\mathbf{A}_2(\boldsymbol{\theta}_2)t}\mathbf{x_1}(t_1, \boldsymbol{\theta}_1),\\
\mathbf{y_2} : \mathbb{R}_+ \to \mathbb{R}^m, \mathbf{y_2}(t, \boldsymbol{\theta}) &= \mathbf{C_2}(\boldsymbol{\theta_2})\mathbf{x_2}(t, \boldsymbol{\theta}),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{x}}_\mathbf{2}(t, \boldsymbol{\theta}) &= \begin{cases} \mathbf{x_2}(t - t_1, \boldsymbol{\theta}) & \forall t \in [t_1, \infty), \\ \mathbf{0} & \forall t \in [0, t_1), \end{cases}\\
\tilde{\mathbf{y}}_\mathbf{2}(t, \boldsymbol{\theta}) &= \mathbf{C_2}(\boldsymbol{\theta_2})\tilde{\mathbf{x}}_\mathbf{2}(t, \boldsymbol{\theta}) \quad \forall t \in [0, \infty),
\end{aligned}$$

allows expression of the output of the LSS as

$$\mathbf{y}(\cdot, \boldsymbol{\theta}) = \mathbf{y_1}(\cdot, \boldsymbol{\theta_1})\left(H_0 - H_{t_1}\right) + \tilde{\mathbf{y}}_\mathbf{2}(\cdot, \boldsymbol{\theta}). \tag{10}$$

2.5.1 Tests of identifiability for LTI systems

The similarity transform approach has been widely used, see for example Godfrey and Chapman [6]. This approach requires a system which is structurally minimal, a consequence of the structural properties of controllability (or reachability) and observability.

As noted previously, a LTI compartmental system is not necessarily structured. In this case it seems reasonable to use a test which does not require the system to have structural properties.

The Laplace Transfer approach, used by Vajda and Rabitz to test a linear, time-invariant and compartmental system for deterministic identifiability does not require the system to have any particular structural properties [9]. For that reason, this approach is suitable for the system of interest of this paper.

2.5.2 The Laplace transform approach

Consider an uncontrolled positive linear time invariant system defined for time set T with $X = \mathbb{R}_+^n$, $Y = \mathbb{R}_+^k$.

It is stated that the conditions $\mathbf{y}(t, \boldsymbol{\theta}) = \mathbf{y}(t, \boldsymbol{\theta}')\ \forall t \in T$ are equivalent to

$$\mathcal{L}\{\mathbf{y}(\cdot, \boldsymbol{\theta})\}(s) = \mathcal{L}\{\mathbf{y}(\cdot, \boldsymbol{\theta}')\}(s)\ \forall s \in \mathbb{C}_0, \tag{11}$$

for $\mathbb{C}_0$ a suitable subset of $\mathbb{C}$. Testing for identifiability requires finding (feasible) solutions for $\boldsymbol{\theta}'$ in equation 11.

The Laplace transform of component y_i of $\mathbf{y}$ has the form

$$\mathcal{L}\{y_i(\cdot, \boldsymbol{\theta})\}(s) = \frac{\phi_{2n}^i(\boldsymbol{\theta})s^{n-1} + \ldots + \phi_n^i(\boldsymbol{\theta})}{s^n + \phi_{n-1}^i s^{n-1}(\boldsymbol{\theta}) + \ldots + \phi_0^i(\boldsymbol{\theta})}. \tag{12}$$

Although T is not stated, the form of equation (12) suggests $T = \mathbb{R}_+$. Suppose $|y_i(t, \boldsymbol{\theta})| \leq Ke^{\lambda t}\ \forall t \in \mathbb{R}_+$ for some constants K, λ, then $\mathcal{L}\{y_i(\cdot, \boldsymbol{\theta})\}$ exists $\forall s \in \mathcal{D}_\lambda$.

The expression for $\mathcal{L}\{y_i(\cdot, \boldsymbol{\theta})\}(s)$ is written in a canonical form by cancelling any common factors between the numerator and denominator and ensuring the coefficient of the highest power of s in the denominator is 1.

Define a vector of observational parameters $\boldsymbol{\phi}(\boldsymbol{\theta})$ by putting $\mathcal{L}\{y_i(\cdot, \boldsymbol{\theta})\}(s)$, $i = 1, \ldots, m$ into canonical form and collecting the coefficients of s in these terms.

2.6 A test case

Consider an uncontrolled parametric linear switching system after equation (5) with $X = \mathbb{R}_+^2$, $Y = \mathbb{R}_+$ and

$$\begin{aligned} \boldsymbol{\theta_1} &= (k_a, k_d, \beta_1)^\top \in \bar{\mathbb{R}}_+^3, \\ \boldsymbol{\theta_2} &= (k_d)^\top \in \bar{\mathbb{R}}_+^1, \\ \boldsymbol{\theta} = \boldsymbol{\theta}_1 \in \Theta &= \bar{\mathbb{R}}_+^3, \\ \alpha_1 &> 0, \\ \mathbf{A_1}(\boldsymbol{\theta_1}) &= \begin{bmatrix} -k_a\alpha_1 & k_d \\ k_a\alpha_1 & -k_d \end{bmatrix}, \\ \mathbf{A_2}(\boldsymbol{\theta_2}) &= \begin{bmatrix} 0 & k_d \\ 0 & -k_d \end{bmatrix}, \\ \mathbf{C_1} = \mathbf{C_2} &= [0\ 1], \\ \mathbf{x_0}(\boldsymbol{\theta_1}) &= (\beta_1, 0)^\top. \end{aligned} \tag{13}$$

As this system represents observations of a chemical system, it is appropriate to assume that rate constants k_a,k_d and initial amount of ligand β_1 have positive values. Further, α_1 is a concentration of reacting species chosen for the experiment. In this simple system α_1 is taken as strictly positive as $\alpha_1 = 0$ means that no reaction can occur.

3 Identifiability of the uncontrolled two phase LSS

Consider an uncontrolled LSS of two phases with response represented by equation (10). Denote the largest eigenvalue of $\mathbf{A_1}$ and $\mathbf{A_2}$ by k_1 and k_2 respectively. Define for $i = 1, 2$
$\mathbf{K_i}(\boldsymbol{\theta_i}) = \mathbf{C_i}(\boldsymbol{\theta_i})(s\mathbf{I}_n - \mathbf{A_i}(\boldsymbol{\theta_i}))^{-1}$. Then
$\mathcal{L}\{\mathbf{y_1}(\cdot, \boldsymbol{\theta_1})\}(s) = \mathbf{K_1}(\boldsymbol{\theta_1})\mathbf{x_0}(\boldsymbol{\theta_1})$, $\forall s \in \mathcal{D}_{k_1}$ and
$\mathcal{L}\{\tilde{\mathbf{y}}_\mathbf{2}(\cdot, \boldsymbol{\theta})\}(s) = \mathbf{K_2}(\boldsymbol{\theta_2})e^{-t_1 s}\mathbf{x_1}(t_1, \boldsymbol{\theta_1}) \forall s \in \mathcal{D}_{k_2}$.

For $t \geq t_1$,

$$\begin{aligned} \mathbf{y_1}(t, \boldsymbol{\theta_1}) &= \mathbf{C_1}(\boldsymbol{\theta_1})e^{\mathbf{A_1}(\boldsymbol{\theta_1})t}\mathbf{x_0}(\boldsymbol{\theta_1}) \\ &= \mathbf{C_1}(\boldsymbol{\theta_1})e^{\mathbf{A_1}(\boldsymbol{\theta_1})(t-t_1)}\mathbf{x_1}(t_1, \boldsymbol{\theta_1}), \end{aligned}$$

then $\forall s \in \mathcal{D}_{k_1}$

$$\mathcal{L}\left\{\mathbf{y_1}(\cdot, \boldsymbol{\theta_1})H_{t_1}\right\}(s) = \mathbf{K_1}(\boldsymbol{\theta_1})e^{-t_1 s}\mathbf{x_1}(t_1, \boldsymbol{\theta_1}).$$

If κ is the largest value of k_1 and k_2, applying the Laplace transform to equation (10) gives

$$\begin{aligned} \mathcal{L}\{\mathbf{y}(\cdot, \boldsymbol{\theta_1})\}(s) &= \mathbf{K_1}(\boldsymbol{\theta_1})(\mathbf{x_0}(\boldsymbol{\theta_1}) - e^{-t_1 s}\mathbf{x_1}(t_1, \boldsymbol{\theta_1})) \\ &+ \mathbf{K_2}(\boldsymbol{\theta_2})e^{-t_1 s}\mathbf{x_1}(t_1, \boldsymbol{\theta_1}) \ \forall s \in \mathcal{D}_\kappa. \end{aligned} \tag{14}$$

The appeal of the Laplace transform method for a linear time invariant system is that from the Laplace transform of output it is easy to obtain relations between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ for the test for identifiability. This feature is not apparent in this setting.

Considering equation (14) for the test case does not illuminate the problem any further. It is quite possible that the field of complex analysis could provide some means of acquiring relationships between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$.

In the absence of any solution method for the equations (11), an alternative approach to the problem is needed.

3.1 An alternative approach

Rather than deal with equation (14) directly, consider the dynamics of the individual phases.

Define the system $M_1(\boldsymbol{\theta_1})$ defined for $T = \mathbb{R}_+$ which has state and output $\mathbf{x_1}$ and $\mathbf{y_1}$ respectively. This system contains the dynamics of the first phase of the original switching system defined for $T = \mathbb{R}_+$.

The original system means that knowledge of the first phase response $\mathbf{y_1}$ is limited to all $t \in [0, t_1)$. By the theory of analytic continuations, this is equivalent to knowledge of $\mathbf{y_1}$ for all $t \in \mathbb{R}_+$. The Laplace transform of the output of this system gives

$$\mathcal{L}\{\mathbf{y_1}(\cdot, \boldsymbol{\theta_1})\}(s) = \mathbf{K_1}(\boldsymbol{\theta_1})\mathbf{x_0}(\boldsymbol{\theta_1}) \ \forall s \in \mathcal{D}_{k_1}.$$

Define the system $M_2(\boldsymbol{\theta})$ which has state and output $\mathbf{x_2}$ and $\mathbf{y_2}$ respectively. This system contains the dynamics of the second phase of the original switching system defined for $t \in [t_1, \infty)$ with initial condition $\mathbf{x_2}(0) = \mathbf{x_1}(t_1)$. For this system

$$\mathcal{L}\{\mathbf{y_2}(\cdot, \boldsymbol{\theta})\}(s) = \mathbf{K_2}(\boldsymbol{\theta_2})\mathbf{x_1}(t_1, \boldsymbol{\theta_1}) \ \forall s \in \mathcal{D}_{k_2}.$$

Assume that $\mathcal{L}\{\mathbf{y_1}(\cdot, \boldsymbol{\theta}_1)\}$ and $\mathcal{L}\{\mathbf{y_2}(\cdot, \boldsymbol{\theta})\}$ are put into canonical form to yield observational parameter vectors $\boldsymbol{\phi}_1(\boldsymbol{\theta}_1)$ and $\boldsymbol{\phi}_2(\boldsymbol{\theta})$ respectively.

3.2 Application to the particular case

Consider the particular system M of equation (13) with response $y(\cdot, \boldsymbol{\theta})$. Consideration of M_1 leads to

$$\begin{aligned} \mathcal{L}\{y_1(\cdot, \boldsymbol{\theta})\}(s) &= \frac{\phi_1(\boldsymbol{\theta})}{s^2 + \phi_2(\boldsymbol{\theta})s} \ \forall s \in \mathcal{D}_0, \\ \phi_1(\boldsymbol{\theta}) &= k_a\alpha_1\beta_1, \\ \phi_2(\boldsymbol{\theta}) &= k_a\alpha_1 + k_d. \end{aligned} \tag{15}$$

As the numerator of $\mathcal{L}\{y_1\}(s)$ is constant, there are no common factors in the numerator and denominator.

As the denominator is also monic, this expression is in the canonical form.

Suppose $\mathbf{x_1}(t_1, \boldsymbol{\theta}) = (x_{1_1}(t_1, \boldsymbol{\theta}), x_{1_2}(t_1, \boldsymbol{\theta}))^\top$. Then

$$\begin{aligned} \mathcal{L}\{y_2(\cdot, \boldsymbol{\theta})\}(s) &= \frac{\phi_3(\boldsymbol{\theta})}{s + \phi_4(\boldsymbol{\theta})} \, \forall s \in \mathcal{D}_{-k_d}, \\ \phi_3(\boldsymbol{\theta}) &= x_{1_2}(t_1, \boldsymbol{\theta}), \\ \phi_4(\boldsymbol{\theta}) &= k_d. \end{aligned} \tag{16}$$

As for $\mathcal{L}\{y_1\}(s)$, $\mathcal{L}\{y_2\}(s)$ is in the required canonical form.

Define $\boldsymbol{\phi}(\boldsymbol{\theta}) = (\phi_1(\boldsymbol{\theta}), \phi_2(\boldsymbol{\theta}), \phi_3(\boldsymbol{\theta}), \phi_4(\boldsymbol{\theta}))$ for this system using equations (15) and (16).

The lurking question is whether the elements of $\boldsymbol{\phi}(\boldsymbol{\theta})$ in fact represent observational parameters. In considering output of M_1, $\boldsymbol{\theta} \in \bar{\mathbb{R}}^3_+$ and $\alpha_1 > 0$ hence $\phi_1(\boldsymbol{\theta})$, $\phi_2(\boldsymbol{\theta})$ are strictly non-zero $\forall \boldsymbol{\theta} \in \Theta$ and are determinable from y_1.

For system M_2, $\phi_4(\boldsymbol{\theta}) \neq 0 \ \forall \boldsymbol{\theta} \in \Theta$ but it is particularly important to consider $\phi_3(\boldsymbol{\theta})$. If $x_{1_2}(t_1, \boldsymbol{\theta}) = 0$, the response y_2 is zero for all time and ϕ_4 is not obtainable from experimental output. Assuming switching time t_1 is sufficiently large to allow some interaction of the chemical species (optical biosensors show response within seconds of starting an experiment,) $\phi_3(\boldsymbol{\theta}) \neq 0$.

For this particular system, note $x_{1_2}(t_1, \boldsymbol{\theta}) = y_1(t_1, \boldsymbol{\theta})$. Knowledge of this value is contained within knowledge of $y_1(t, \boldsymbol{\theta})$, $\forall t \in [0, t_1)$ and hence no additional information on the system beyond $\boldsymbol{\phi}_1$ is provided by ϕ_3. Hence it is not necessary to determine the explicit expression for $x_{1_2}(t_1, \boldsymbol{\theta})$.

Solving the equations of the identifiability test shows $ID(M, \boldsymbol{\phi}) = \{\boldsymbol{\theta}\}$ for all $\boldsymbol{\theta} \in \Theta$ and the system of equation (13) is deterministically globally identifiable.

4 Conclusions

The methodology of this paper has shown that under a definition in the spirit of the existing theory, a test case LSS is classifiable as deterministically globally identifiable. This classification is made without having to obtain explicit expressions for all elements of $\boldsymbol{\phi}$. Although this allows a clear result in this paper, it effectively truncates the exploration of this methodology. A suitable test case may show if the technique is viable when it is necessary to find the symbolic expression for the state variable of the first phase.

To embrace a wider range of applications, the methodology presented here could be broadened in a later paper.

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