

Analysis and Numerical Approximation of a Singular Boundary Value Problem for the One-dimensional p-Laplacian

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Abstract: In this work we are concerned about a singular boundary value problem involving the p-laplacian which arises in mathematical models of fluid mechanics. We analyze the asymptotic behavior of the solutions of the considered ordinary differential equation near the singularities and introduce a computational method which takes this behavior into account.

Key-Words: Singular boundary value problem, p-laplacian, upper and lower solutions, one-parameter families of solutions, asymptotic expansions.

1 Introduction

The mathematical modelling of various phenomena of mechanics and physics leads to the following Dirichlet problem:

$$\begin{aligned} -\operatorname{div} (|\nabla g|^{p-2} \nabla g) &= f(|x|, g), & x \in B; \\ g(x) &> 0, & x \in B; \\ g(x) &= 0, & x \in \partial B, \end{aligned} \tag{1}$$

where B is the unit ball centered at the origin in \mathbb{R}^N , $N > 1$, $p > 1$.

The nonlinear differential operator in the left-hand side of (1) is known as the p-laplacian and, in the case of $p = 2$, reduces to the usual laplacian.

It is worth to remark that the laplacian operator arises in the modeling of many classical problems of fluid dynamics, as the result of the application of a "linear flow law" (see [1]). For example, in the case of heat conduction problems, if we denote by g the steady state temperature, according to the Fourier's law, the heat flow vector is proportional, in module, to $|\nabla g|$. When this law is applicable, the heat diffusion is given by $\operatorname{div}(\nabla g)$, which yields the laplacian operator. However, in the modeling of certain processes, for example, the melting of ice at a constant temperature, the heat flow is better described by a generalized Fourier's law, which states that the flow vector is a nonlinear function of the temperature gradient, with the form $|\nabla g|^{p-2} \nabla g$ (see, for example, [2]). Then the heat diffusion will be represented by the p-laplacian.

Similar situations, when the generalization of classical linear problems leads to nonlinear ones, in-

volving the p-laplacian, also occur when modeling flows of non-newtonian fluids [1], and in elasticity theory, when modelling the deformation of nonlinear membranes [3].

One is often interested in radial solutions of problem (1), that is, solutions that depend only on $|x|$. In that case, problem (1) reduces to a boundary value problem for an ordinary differential equation:

$$\begin{aligned} -r^{1-N} (r^{N-1} |g'|^{p-2} g')' &= f(r, g), & r \in (0, 1); \\ g(r) &> 0, & r \in (0, 1); \\ g'(0) &= 0, g(1) = 0, \end{aligned} \tag{2}$$

where $N \geq 1$ is the space dimension. We shall assume that $f : (0, 1) \times (0, \infty) \rightarrow \mathbb{R}$ is continuous in the considered domain and has singularities at $y = 0$ or $r = 0$.

The existence and the properties of solutions to problem (2), under different assumptions on the function f , were investigated by many authors. The existence and uniqueness of solution in the case $p = 2$, $N = 1$, with $f(r, g) = rg^n, n < 0$ was proved by Nachman and Callegari [4], in connection with a boundary layer problem in fluid mechanics. In [5],[6] and [7] the authors also considered the case $p = 2$, but with $f(r, g) = r^\sigma g^n, \sigma > -1$. In these works the asymptotic behavior of the solutions near the singularity at $r = 1$ has been analyzed and numerical methods were introduced, which take into account this behavior. Moreover, upper and lower solutions were determined and, in some particular cases, an explicit formula for the exact solution was obtained.

Concerning the case $p \neq 2$, in [8] and [9] Wong has obtained sufficient conditions for existence and uniqueness of solutions of problem (2), for $N = 1$.

In a recent work, Jin, Yin and Wang [10], using the method of lower and upper solutions, have investigated the existence of solutions to problem (2) in the general case $N \geq 1$.

In the present paper, we will continue the investigation of problem (2) in the case $N = 1$, for $p > 1$ and $f(r, g) = ar^\sigma g^n$, where a is a negative constant. The values of σ and n for which the existence of solution is guaranteed depend on p . According to the Existence Theorem of [8], a sufficient condition for the existence of solution is $\sigma \geq 0, n - p + 1 < 0, p \geq 2$. On the other hand, uniqueness of solution in this case is guaranteed by Theorem 2.3 of [9]. As we shall see in the next section, existence of solution can be guaranteed under weaker conditions.

When $n < 0$ this problem has a singularity at $r = 1$, since the second derivative of g is not continuous at this point. The first derivative has also a discontinuity at this point, if $n < -1$. Moreover, the problem can have also a singularity at $r = 0$, for some values of σ and p . This will be discussed in the next sections.

2 Behavior of the solution in the neighborhood of $r = 0$

Consider the singular Cauchy problem:

$$(|g'(r)|^{p-2} g'(r))' = ar^\sigma g^n(r), \quad 0 < r < r_0, \quad (3)$$

$$\lim_{r \rightarrow 0^+} g'(0) = 0. \quad (4)$$

In this section we will analyze the asymptotic behavior of the solutions of this Cauchy Problem when $r \rightarrow 0^+$. Our main result is resumed in the following proposition.

Proposition 1 Assume that in (3) $p > 2, \sigma \geq 0, n < 0$ and $r_0 > 0$. The Cauchy problem (3)-(4) has, in the neighborhood of the singular point $r = 0$, a one parameter family of solutions that can be represented by:

$$g_1(r, b) = C_1 - br^k \left[1 - \frac{bn(1 + \sigma)}{2C_1(2p + 2\sigma - 1)} r^k + o(r^k) \right], \quad (5)$$

where $k = \frac{p+\sigma}{p-1}, C_1 = \left(\frac{-a(p-1)^{p-1}}{b^{p-1}(p+\sigma)^{p-1}(\sigma+1)} \right)^{-\frac{1}{n}}$ and $b > 0$ is the parameter.

Proof. In the neighborhood of $r = 0$, let us look for a solution of (3)–(4) in the form

$$\begin{aligned} g(r) &= C_1 - C_2 r^k [1 + o(1)] \\ g'(r) &= -kC_2 r^{k-1} [1 + o(1)], \\ g''(r) &= -k(k-1)C_2 r^{k-2} [1 + o(1)], \quad r \rightarrow 0^+. \end{aligned} \quad (6)$$

Substituting in (3) we obtain

$$\lim_{r \rightarrow 0^+} k^{p-1} (k-1) (p-1) C_2^{p-1} r^{k(p-1)-p-\sigma} C_1^{-n} \times \left[1 - \frac{C_2}{C_1} r^k \right]^{-n} = a, \quad (7)$$

which implies $k = \frac{p+\sigma}{p-1}, k-1 = \frac{\sigma+1}{p-1}$, and with $C_2 = b > 0$ we have $C_1 = \left(\frac{-a(p-1)^{p-1}}{b^{p-1}(p+\sigma)^{p-1}(\sigma+1)} \right)^{-\frac{1}{n}}$.

Defining $y(r)$ by $g(r) = C_1 - C_2 r^k [1 + y(r)]$, we obtain the Cauchy problem in y

$$\begin{aligned} &\frac{(p-1)^2}{\sigma+1} [1 + y + \frac{r}{k} y']^{p-2} \times \\ &[(k-1)k(1+y) + 2kry' + r^2 y''] = \\ &\left[1 - \frac{b}{C_1} r^k (1+y) \right]^n, \end{aligned} \quad (8)$$

$$y(0) = \lim_{r \rightarrow 0^+} ry'(r) = 0, \quad (9)$$

where $r = 0$ is a regular singular point of (8).

For each $b \neq 0$, this problem has a particular solution that can be represented by

$$y_{par}(r, b) = \sum_{l=0, j=0, l+j \geq 1}^{+\infty} y_{l,j}(b) r^{l+j \frac{p+\sigma}{p-1}},$$

where $0 < r \leq \delta(b), \delta(b) \geq 0$ and the coefficients $y_{l,j}$, depending on b , may be determined substituting y_{par} in (8), which gives in the case $l = 0, j = 1$

$$y_{0,1} = -\frac{bn(1 + \sigma)}{2C_1(2p + 2\sigma - 1)}.$$

Writing out the leading linear homogeneous terms of (8) for solutions that satisfy (9) we obtain the equation

$$r^2 y'' + \frac{3p + p\sigma - 2}{p-1} r y' + \frac{(p + \sigma)(\sigma + 1)}{p-1} y = 0, \quad r \rightarrow 0^+$$

whose characteristic exponents are $\lambda_1 = -\frac{p+\sigma}{p-1} < 0$ and $\lambda_2 = -(\sigma + 1) < 0$. Therefore, for each $b \neq 0$ this problem has no other solution than $y_{par}(r, b)$. ■

From Proposition 1 we can conclude that if g is a solution of problem (3)-(4), then g'' will have a discontinuity at $r = 0$ whenever $k < 2$, that is, when $\sigma < p - 2$.

3 The singularity at $r = r_0$

In order to obtain the behavior of the solution in the neighborhood of the singular point $r = r_0$, let us now consider the singular Cauchy problem:

$$(|g'(r)|^{p-2} g'(r))' = ar^\sigma g^n(r), \quad 0 < r < r_0, \quad (10)$$

$$g(r_0) = \lim_{r \rightarrow r_0^-} [(r_0 - r) g'(r)] = 0. \quad (11)$$

As we shall see, the asymptotic behavior of the solutions when $r \rightarrow r_0^+$ may change significantly, depending on the value of n (as it happens in the case $p = 2$). Here we will focus on the two main cases: $-1 < n < 0$ and $n < -1, n \neq -2 - \frac{1}{p-1}$. The cases of $n = -1$ and $n = -2 - \frac{1}{p-1}$ will not be treated here. Details about these cases can be found in [11].

Proposition 2 Assume that in (10) $p > 1, \sigma \geq 0, n < 0$ and $r_0 > 0$. The Cauchy problem (10)-(11) has, in the neighborhood of the singular point $r = r_0$, a one parameter family of solutions that can be represented by:

$$g(r, c) = \begin{cases} g_2(r, c), & \text{if } n < -1 \quad n \neq -2 - \frac{1}{p-1}; \\ g_3(r, c), & \text{if } n > -1; \end{cases} \quad (12)$$

as $r \rightarrow r_0^-$, where c is the parameter,

$$g_2(r, c) = \left(\frac{ar_0^\sigma (p-1-n)^p}{p^{p-1}(p-1)(1+n)} \right)^{\frac{1}{p-1-n}} (r_0 - r)^{\frac{p}{p-1-n}} \left[1 - \frac{(1+n)p\sigma}{(p-1-n)(2p+np-n-1)r_0} (r_0 - r) + c (r_0 - r)^{\frac{-p(1+n)}{p-1-n}} + O((r_0 - r)^{1+\mu}) \right], \quad (13)$$

$$\mu = \min \left\{ 1, -\frac{p(1+n)}{p-1-n} \right\};$$

$$g_3(r, c) = \left(\frac{ar_0^\sigma}{(p-1)(2+n)(1+n)c} \right)^{\frac{1}{p-2-n}} (r_0 - r) + c (r_0 - r)^{2+n} \left[1 + y_{0,1}(c) (r_0 - r)^{1+n} + o\left((r_0 - r)^{1+n}\right) \right], \quad (14)$$

Proof. First, consider the case $n < -1$. It can be easily proved that in the neighborhood of $r = r_0$, a solution of problem (10)-(11) may be given by

$$g(r) = C (r_0 - r)^k [1 + o(1)], \quad r \rightarrow r_0^-, \quad (15)$$

where $k = \frac{p}{p-1-n} > 0, k(k-1) = \frac{p(1+n)}{(p-1-n)^2} < 0$ and $C = \left(\frac{ar_0^\sigma (p-1-n)^p}{p^{p-1}(p-1)(1+n)} \right)^{\frac{1}{p-1-n}}$.

If in (10)-(11) we perform the variable substitution

$$g(r) = C (r_0 - r)^k [1 + y(r)], \quad (16)$$

we obtain the Cauchy problem in y

$$\begin{aligned} & \frac{(p-1-n)^2}{p(n+1)} \left[1 + y - \frac{p-n-1}{p} (r_0 - r) y' \right]^{p-2} \\ & \left[(r_0 - r)^2 y'' - \frac{2p}{p-1-n} (r_0 - r) y' \right. \\ & \left. + \frac{p(1+n)}{(p-1-n)^2} (1+y) \right] - \left(\frac{r}{r_0} \right)^\sigma (1+y)^n = 0, \end{aligned} \quad (17)$$

$$0 < r < r_0$$

$$y(r_0) = \lim_{r \rightarrow r_0^-} [(r_0 - r) y'(r)] = 0, \quad (18)$$

where $r = r_0$ is a regular singular point of equation (17).

Writing out the leading linear homogeneous terms of (17), for solutions that satisfy (18), we obtain the equation

$$(r_0 - r)^2 y'' - \frac{2p+(p-2)(n+1)}{p-1-n} (r_0 - r) y' + \frac{p(1+n)}{p-1-n} y = 0, \quad u \rightarrow r_0^- \quad (19)$$

whose characteristic exponents are $\lambda_1 = -1 < 0$ and $\lambda_2 = -\frac{p(1+n)}{p-1-n} > 0$. When $n = -2 - \frac{1}{p-1}$, we obtain $\lambda_2 = 1$, and therefore the difference between the characteristic exponents is an integer. In this case the expansion of y has a different form and it is out of scope of the present work. For any other value of $n < -1$ the problem (17)-(18) has a particular solution, $y_{par}(r)$, holomorphic in $r = r_0$ and if $r \neq r_0$,

$$y_{par}(r) = \sum_{l=1}^{+\infty} y_l (r_0 - r)^l, \quad |r_0 - r| \leq \delta, \quad \delta > 0,$$

where the coefficients y_l may be determined substituting $y_{par}(r)$ in (17). For example, with $l = 1$, we get

$$y_1 = -\frac{(1+n)p\sigma}{2(p-1-n)(2p+np-n-1)r_0}.$$

The one-parameter family of solutions of (17)-(18) which reduces to y_{par} when the parameter is zero, can be represented by:

$$y(r, c) = y_1 (r_0 - r) + O\left((r_0 - r)^2\right) + c (r_0 - r)^{\lambda_2} [1 + o(1)], \quad (20)$$

when $r \rightarrow r_0^-$. Moreover, taking into account that (17) is an asymptotically autonomous equation, this family may be represented in the form of a convergent series:

$$\begin{aligned} y(r, c) = & y_{par}(r) + c (r_0 - r)^{\lambda_2} [1 + b_1 (r_0 - r) + \\ & b_2 (r_0 - r)^2 + \dots] + c^2 (r_0 - r)^{2\lambda_2} \\ & [c_0 + c_1 (r_0 - r) + c_2 (r_0 - r)^2 + \dots] + \dots, \end{aligned} \quad (21)$$

$$|r_0 - r| \leq \Delta(c), \quad \Delta(c) > 0,$$

and the one-parameter family of solutions of the Cauchy problem (10)–(11) that we were looking for, in the case $n < -1$ and $n \neq -2 - \frac{1}{p-1}$, is given by (13).

Let us now consider the case $n > -1$. In the neighborhood of $r = r_0$, we shall look for a solution of this problem in the form:

$$\begin{aligned} g(r) &= C_1 (r_0 - r) + C_2 (r_0 - r)^k [1 + o(1)], \\ g'(r) &= -C_1 - kC_2 (r_0 - r)^{k-1} [1 + o(1)], \\ g''(r) &= k(k-1)C_2 (r_0 - r)^{k-2} [1 + o(1)], \quad r \rightarrow r_0^- . \end{aligned}$$

From (10)-(11),

$$(p+1)k(k-1)C_2 (r_0 - r)^{k-2-n} C_1^{p-n} |_{r \rightarrow r_0^-} = ar_0^\sigma,$$

which implies $k = 2 + n > 0$, $k - 1 = n + 1 > 0$ and with $C_2 = c < 0$ we get $C_1 = \left(\frac{ar_0^\sigma}{(p+1)(2+n)(1+n)c} \right)^{\frac{1}{p-n}}$.

Next, we define the function $y(r)$ by

$$g(r) = C_1 (r_0 - r) + c (r_0 - r)^k [1 + y(r)].$$

Substituting in (10)-(11), we obtain the Cauchy problem in y

$$\begin{aligned} &\left[1 + \frac{c}{C_1} k (r_0 - r)^{k-1} - \frac{c}{C_1} (r_0 - r)^k y' \right]^p \times \\ &\left[(r_0 - r)^2 y'' - 2k (r_0 - r) y' + k(k-1)(1+y) \right] \\ &= k(k-1) \left(\frac{r}{r_0} \right)^\sigma \left[1 + \frac{c}{C_1} (r_0 - r)^{k-1} (1+y) \right]^n, \\ &0 < r < r_0, \end{aligned} \tag{22}$$

$$y(r_0) = \lim_{r \rightarrow r_0^-} [(r_0 - r) y'(r)] = 0, \tag{23}$$

where $r = r_0$ is a regular singular point of (22)–(23).

For any $c \neq 0$, this problem has a particular solution that can be represented by the convergent series

$$y_{par}(r, c) = \sum_{l=0, j=0, l+j \geq 1}^{+\infty} y_{l,j}(c) (r_0 - r)^{l+j(1+n)},$$

where $0 \leq r_0 - r \leq \delta(c)$, $\delta(c) \geq 0$ and the coefficients $y_{l,j}$ may be determined substituting y_{par} in (22).

If we write the leading linear homogeneous terms of (22) we obtain

$$\begin{aligned} (r_0 - r)^2 y'' - 2(2+n)(r_0 - r) y' + \\ (2+n)(n+1) y = 0, \end{aligned} \tag{24}$$

as $r \rightarrow r_0^-$. This equation has the characteristic exponents $\lambda_1 = -1 - n < 0$ and $\lambda_2 = -2 - n < 0$.

Then (22)–(23) has no other solution than $y_{par}(r, c)$. Summarizing, we conclude that, for any $-1 < n < 0$, the Cauchy problem (10)–(11) has a one-parameter family of solutions that can be represented by (14).

■

From Proposition 2, some conclusions can be taken concerning the singularity of the problem (10),(11) at $r = r_0$. When $-1 < n < 0$, g' is continuous at $r = r_0$, but g'' isn't. When $n < -1$, the first derivative is also discontinuous at this point. The same happens when $n = -1$ (details about this case can be found in [11]).

4 Lower and Upper Solutions

Let us consider again the boundary value problem

$$(|g'(r)|^{p-2} g'(r))' = ar^\sigma g^n(r), \quad 0 < r < r_0 \tag{25}$$

$$\lim_{r \rightarrow 0^+} g'(0) = 0, \tag{26}$$

$$\lim_{r \rightarrow r_0^-} g(r) = 0, \tag{27}$$

with $p > 1$, $\sigma > -1$ and $n < -1$.

Definition 3 We shall say that $h(r)$ is a lower (resp. upper) solution of the problem (25)-(27) if $h(r) \in C[0, r_0] \cap C^2(0, r_0)$ and satisfies

$$\begin{aligned} -(|h'(r)|^{p-2} h'(r))' + ar^\sigma h^n(r) \leq 0 \text{ (resp. } \geq 0), \\ 0 < r < r_0, \quad h'(0) \leq 0 \text{ (resp. } \geq 0), \quad h(r_0) = 0. \end{aligned}$$

We want upper and lower solutions to verify

$$\begin{aligned} \lim_{r \rightarrow 0^+} h'(r) &= 0, \\ \lim_{r \rightarrow r_0^-} h(r) &= 0, \\ \lim_{r \rightarrow r_0^-} \frac{h(r)}{g(r)} &= \alpha_1 \neq 0, \end{aligned}$$

where the last condition means that the upper and lower solutions are asymptotically equivalent to the exact solution in the neighborhood of the singular point $r = r_0$. Taking into account the asymptotic behavior of the solutions at $r = 0$ and $r = r_0$, studied in the previous sections, we will look for lower and upper solutions in the form

$$h(r) = B \left(r_0^k - r^k \right)^{\frac{p}{p-1-n}}, \tag{28}$$

where $k = \frac{\sigma+p}{p-1}$ and B is a positive constant. Note that in the case $p = 2$ we obtain the upper and lower solutions, introduced in [7]. We have $k > 1$, which follows from $\sigma > -1$ and $p > 1$. We want to define

B in such a way that h satisfies equation (25), that is, h must verify

$$(|h'(r)|^{p-2}h'(r))' = ah(r)^n r^\sigma \quad (29)$$

With this purpose, we note that

$$\begin{aligned} & (|h'(r)|^{p-2}h'(r))' = \\ & = \frac{1}{Bkp}(-1+p)r^\sigma(r_0^k - r^k)^{\frac{pn}{p-1-n}} \left(\frac{Bkp}{p-1-n}\right)^p q(r) \end{aligned} \quad (30)$$

where $q(r) = -(p-1-n)(k-1)r_0^k + (1+n+(k-1)p)r^k, r \in [0, r_0]$.

On the other hand, from (28) we have

$$h(r)^n = B^n \left(r_0^k - r^k\right)^{\frac{pn}{p-1-n}} \quad (31)$$

By substituting (30) and (31) into (29) we obtain

$$-\frac{1}{Bkp}(-1+p) \left(\frac{Bkp}{p-1-n}\right)^p q(r) + aB^n = 0. \quad (32)$$

Note that if

$$1+n+(k-1)p = 0 \quad (33)$$

then $q(r)$ reduces to a constant:

$$q(r) \equiv r_0^k(k-1)(1+n-p). \quad (34)$$

In this case, that is, if $n = (1-k)p - 1$, equation (32) takes the form

$$-\frac{1}{Bkp}(-1+p) \left(\frac{Bkp}{p-1-n}\right)^p r_0^k(k-1)(1+n-p) + aB^n = 0 \quad (35)$$

Solving (35) with respect to B , we obtain

$$B = \left(\frac{-a(p-1-n)^{p-1}}{r_0^k(kp)^{p-1}(k-1)}\right)^{\frac{1}{p-1-n}}. \quad (36)$$

Hence we conclude that when the condition (33) is satisfied the exact solution of the problem (25)-(27) has the form (28), with B given by (36). For values of n, p, σ which do not satisfy that condition we cannot find an explicit formula for the exact solution. However, by analyzing the function $q(r)$, we can obtain conditions on B , under which the function h can be an upper or a lower solution, according to Definition 3. This is summarized in the next proposition.

Proposition 4 *Let $\sigma > -1, n < -1, p > 1$ and let $B > 0$ be a constant. Then the function defined by (28) will be*

- a lower solution of the problem (25)-(27) if B satisfies

$$B \leq B_1 = \left(\frac{a(p-1-n)^p(p-1)^{p-1}}{p^{p-1}(1+n)(\sigma+p)r_0^k}\right)^{\frac{1}{p-1-n}},$$

when $n \leq (1-k)p - 1$,

and

$$B \leq B_2 = \left(-\frac{a(p-1-n)^{p-1}(p-1)^{p-1}}{p^{p-1}(\sigma+p)^{p-1}(\sigma+1)r_0^{\frac{\sigma+p}{p-1}}}\right)^{\frac{1}{p-1-n}},$$

when $-1 > n \geq (1-k)p - 1$.

- an upper solution of the problem (25)-(27) if B satisfies

$$B \geq B_1, \quad , \text{ when } -1 > n \geq (1-k)p - 1,$$

and

$$B \geq B_2, \quad , \text{ when } n \leq (1-k)p - 1.$$

We see that when $n = (1-k)p - 1, B_1 = B_2 = B$, where B is given by (36). As we have remarked above, in this case we have an exact solution in the form (28).

The upper and lower solutions can be used to obtain suitable initial approximations for the application of computational methods. On the other hand, we can use them to apply the existence results obtained in [10]. Actually, choosing an upper solution g_1 and a lower solution g_2 of the form (28) the conditions of proposition 2.1 of that work are satisfied and therefore the problem has at least one solution g , such that $g_1 \leq g \leq g_2$. Hence, we conclude that the problem (25)-(27) is solvable for $p > 1, \sigma > -1, n < -1$. Note that the problem is also solvable in the case $-1 \leq n < 0$, but in this case the upper and lower solutions must be sought with a different form.

5 Numerical results

Let us briefly describe the numerical algorithm we used in order to obtain the approximate solutions of problem (25)-(27). When $p > 2 + \sigma$, for given values of a, n, r_0, σ , we have considered two regular Cauchy problems

$$\begin{cases} (|g'(r)|^{p-2}g'(r))' = ar^\sigma g^n(r), & 0 < r < \frac{r_0}{2}, \\ g(\delta) = \bar{g}_1(\delta, b), & g'(\delta) = \bar{g}'_1(\delta, b) \end{cases} \quad (37)$$

$$\begin{cases} (|g'(r)|^{p-2}g'(r))' = ar^\sigma g^n(r), & \frac{r_0}{2} < r < r_0, \\ g(r_0 - \delta) = \bar{g}_i(r_0 - \delta, c), & g'(r_0 - \delta) = \bar{g}'_i(r_0 - \delta, c), \end{cases} \quad (38)$$

which we denote Problem A and Problem B, respectively. Here $i = 2$ or 3 , depending on n, δ is a small constant. The functions \bar{g}_1, \bar{g}_2 and \bar{g}_3 are computed

according to the formulae (5), (13) and (14), respectively, ignoring the remainders of the series. We solve problems *A* and *B* for certain values of the parameters *b* and *c*. With this purpose we use the program ND-Solve of Mathematica. Then, following the idea of the shooting method, we determine the values of *b* and *c* so that *g* and *g'* are continuous at $r = r_0/2$. This gives a system of two nonlinear equations, which is solved by the Newton's method.

When $p \leq 2 + \sigma$, since we have no singularity at $r = 0$, we solve the auxiliary problem (B) starting from $r_0 - \delta$, for a certain value of the parameter *c*. Then we compute *c* by the shooting method, requiring that the boundary condition at $r = 0$ is satisfied.

In our calculations we have used $\delta = 0.001$.

In figures 1,2 we plot some approximate solutions of problem (25)-(27) for different values of σ, n, r_0, a and *p*.

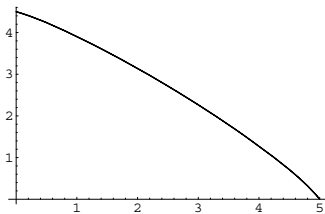


Figure 1: Solution of the boundary value problem with $r_0 = 5, \sigma = -0.5, n = -\frac{5}{3}, a = -2, p = 4$.

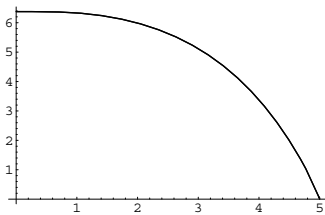


Figure 2: Solution of the boundary value problem with $r_0 = 5, \sigma = 0.5, n = -0.6, a = -1, p = 1.8$.

In table 1 we compare the computed values of the solutions at $r = r_0/2$ with the exact ones, in some cases where the exact solution is known. The obtained numerical results show that the used method has an accuracy of about 8 digits.

References:

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(p, σ, n, r_0, a)	exact value	appr. value
$(1.5, 0, -4, 1, -1)$	0.9564656	0.9564650
$(3, 0, -\frac{5}{2}, 1, -1)$	0.74763296	0.74763297
$(4, 0.5, -3, 5, -2)$	2.62264233	2.62264236
$(5, 2, -\frac{19}{4}, 0.5, -3)$	0.46947811	0.46947813
$(6, 1, -\frac{17}{5}, 2, -1)$	1.16752210	1.16752212
$(4, -0.5, -\frac{5}{3}, 5, -2)$	2.71494726	2.71494720

Table 1: Comparison between the exact and the approximate solution at $r = \frac{r_0}{2}$

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