Robust Controller Designs for Systems with Nonlinear Uncertainties

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Abstract: - In this work, we revisit the robust stability margin optimization problem. Two robust controller designs are proposed for systems subject to bounded nonlinear time-varying uncertainty. The first method is a counterpart of the well known D-K iteration procedure for μ synthesis. The second method is a new one on the basis of a new bilinear matrix inequality (BMI) formulation. Comprehensive comparisons of the two methods are made and a numerical example is given to illustrate the results.

Key-Words: - robust stability, stability multiplier, linear matrix inequality (LMI), nonlinear time-varying uncertainty.

1 Introduction

In 1985 the well known $D-K$ iteration [1] was proposed for designing robust controllers to meet a certain frequency-dependent scaled small gain condition [1] in order to make the systems robust against structured dynamic uncertainties. It has also been shown that the constantly scaled small gain condition is a necessary and sufficient condition for the nominal system to be uniformly robustly stable against the set of bounded nonlinear time-varying uncertainties [2]. While it has been addressed in [3] that the constant multiplier for the nonlinear perturbation case, i.e., the scaling for the scaled small gain condition, can be found via solving over a certain LMI, little attention has been paid toward the synthesis problem except [4,5]. In this paper, we'd like to treat the controller design problem in the passivity framework. New BMI formulation based on the recently developed approach [6] for the synthesis problem will be derived and different robust controller design methods based on iterative LMI (ILMI) method will be given.

The paper is organized as follows. Section 2 gives the formal problem statement and some preliminaries for future developments. Section 3 presents the two proposed approaches for the robust controller synthesis problem. In Section 4, a numerical example is provided to demonstrate the results. Section 5 is the conclusions.

2 Problem Formulation

Notation

Most of the notation and terminology are standard and will be defined as the need arises.

Table 1.

$\mathbb{R}$	the set of real numbers
A^*	the complex conjugate transpose of A
$\lVert G \rVert$	the L_2 gain of a nonlinear time-varying operator
$F_l(\bullet,\bullet)$	denotes the lower linear fraction representation, see [7].
$\underline{\underline{ss}}$	The state-space realization of $H(s)$ is denoted as: $H(s)\ \underline{\underline{ss}} \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$.

Problem Description

The robust control paradigm considered is depicted in Fig. 1.

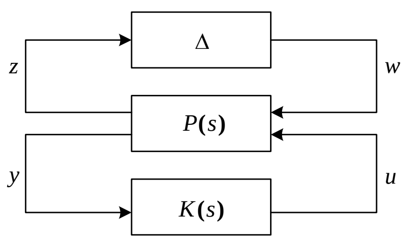

Fig. 1 $\Delta - P - K$ framework

Throughout this paper, we assume the uncertainty Δ belongs to the bounded nonlinear time-varying uncertainty set Δ_{NL} as follows,

$$\Delta_{NL} := \left\{ \begin{array}{l} diag(\delta_1,\ldots,\delta_L), \\ \delta_i \text{ is } a \text{ nonlinear, time-varying operator}, \\ i = 1,\ldots,L \end{array} \right\}.$$

Here, it means that Δ_{NL} can be specified by $w_i = \delta_i z_i$, where each single-input single-output nonlinear time-varying perturbation δ_i has an L_2 gain less than γ. The symbol P denotes the generalized plant, including the nominal plant, weighting functions, etc, described by

$$P \quad \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_P}$, $w \in \mathbb{R}^m$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^m$, and $y \in \mathbb{R}^{n_y}$ with $m := m_1 + \cdots + m_L$. The symbol K denotes a dynamic controller of the form

$$K \quad \begin{cases} \dot{x}_K = A_K x_K + B_K y \\ u = C_K x_K + D_K y \end{cases} \tag{2}$$

where $x \in \mathbb{R}^{n_K}$, to be designed. Our goal is to find a dynamic controller $K(s)$ of form (2) to enlarge the robust stability margin [3], defined as the largest size of the structured uncertainties against which the system is robustly stable, of the perturbed system in Fig. 1. Specifically, for all $\Delta \in \Delta_{NL}$ with $\|\Delta\| \le \gamma$ (i.e., $|\delta_i| \le \gamma$ for all i) we want to design a dynamic controller of form (2) to maximize the value γ such that the closed-loop system (1)-(2) is robustly stable.

By [3] The robust controller synthesis problem we consider is to maximize the value γ via finding a dynamic controller $K(s)$ of form (2) and a generalized stability multiplier $W = diag(W_1,\ldots,W_L)$ where $W_i \in \mathbb{R}$, $i = 1,\ldots,L$ such that $\tilde{M}_\gamma$ is stable and the following conditions

$$\tilde{M}_\gamma(j\omega)W + W\tilde{M}_\gamma(j\omega)^* > 2\varepsilon I, \tag{3}$$

$$W > 0, \tag{4}$$

hold for some $\varepsilon > 0$ and for all $\forall \omega \in \mathbb{R} \cup \{\infty\}$, where $\tilde{M}_\gamma := (I - \gamma M)(I + \gamma M)^{-1} = F_l(\tilde{P}_\gamma, K)$, and the sector transformed plant $\tilde{P}_\gamma = S \bigstar \Gamma P$, where

$$S = \begin{pmatrix} I_m & -\sqrt{2} I_m \\ \sqrt{2} I_m & -I_m \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma I_m & 0 \\ 0 & I_{n_y} \end{pmatrix},$$

and the symbol $\bigstar$ means star product [7]. A tedious algebraic manipulation shows that

$$\tilde{P}_\gamma(s) \underset{=}{\overset{ss}{}} \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array} \right]$$

where

$\tilde{A} = A - \gamma B_1 (I + \gamma D_{11})^{-1} C_1$;

$\tilde{B}_1 = \sqrt{2} B_1 (I + \gamma D_{11})^{-1}$;

$\tilde{B}_2 = B_2 - \gamma B_1 (I + \gamma D_{11})^{-1} D_{12}$;

$\tilde{C}_1 = -\sqrt{2}\gamma (I + \gamma D_{11})^{-1} C_1$;

$\tilde{C}_2 = C_2 - \gamma D_{21} (I + \gamma D_{11})^{-1} C_1$;

$\tilde{D}_{11} = (I + \gamma D_{11})^{-1} (I - \gamma D_{11})$;

$\tilde{D}_{12} = -\sqrt{2}\gamma (I + \gamma D_{11})^{-1} D_{12}$;

$\tilde{D}_{21} = \sqrt{2} D_{21} (I + \gamma D_{11})^{-1}$;

$\tilde{D}_{22} = D_{22} - \gamma D_{21} (I + \gamma D_{11})^{-1} D_{12}$.

3 Main Results

In this section a new BMI formulation for the robust controller design problem mentioned in Section 2 is presented, where the LMI approach developed by Scherer et al [6] for computing a SPR dynamic controller will be used. Accordingly, two ILMI-based robust controller design procedures will be given on the basis of different ways of partitioning the BMI variables.

To proceed, first, denote the set of all constant diagonal generalized multiplier W to be Π^{NL} where $\Pi^{NL} = \{ diag(W_1,\ldots,W_L) | W_i > 0, i = 1,\ldots,L \}$. It follows that the robust controller synthesis conditions: closed-loop stability , (3) and (4), can be reinterpreted as follows.

Theorem 1. Given a positive value γ. The nominal system $F_l(P,K)$ described by (1) and (2) is uniformly robustly stable against the set of uncertainties $\Delta \in \Delta_{NL}$ with size no greater than γ if there exist a controller K and a stability multiplier W satisfying:

(i) $W \in \Pi^{NL}$,

(ii) $F_l(\hat{P}_\gamma, K)(s)$ is strictly positive real,

where $\hat{P}_\gamma(s) = \tilde{P}_\gamma(s) diag(W, I)$.

Next we attemp to adopt the LMI approach developed by Scherer et al [6] for computing the SPR dynamic controller. Since the direct through part "D_{22}" of $\hat{P}_\gamma$ is not necessarily null, a suitable loop transform is required to circumvent the difficulty. This can be done by defining

$$\breve{P}_\gamma := \begin{pmatrix} \tilde{P}_{\gamma_{11}} & \tilde{P}_{\gamma_{12}} \\ \tilde{P}_{\gamma_{21}} & \tilde{P}_{\gamma_{22}} - \tilde{D}_{22} \end{pmatrix} \underset{=}{\overset{ss}{}} \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & 0 \end{array} \right]$$

and

$$K_{eq} := K\left(I - \tilde{D}_{22}K\right)^{-1}.$$

Thus

$F_l(\hat{P}_\gamma, K) = F_l(\bar{P}_\gamma, K_{eq})$, where $\bar{P}_\gamma = \breve{P}_\gamma \cdot diag\left(W, I\right)$, in which the direct through part "D_{22}" of $\bar{P}_\gamma$ is null. Now it's ready to the method by Scherer et al [6], this leads to the following BMI formulation for the synthesis problem: Maximize the value γ subject to the following BMIs (5)-(7), i.e., maximize γ subject to the existence of the matrix variables W, X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ satisfying the following BMIs.

$$W > 0 \quad (5)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \quad (6)$$

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) \\ * & (2,2) & (2,3) \\ * & * & (3,3) \end{bmatrix} < 0 \quad (7)$$

where

$$(1,1) := \tilde{A}X + X\tilde{A}^T + \tilde{B}_2\hat{C} + \hat{C}^T\tilde{B}_2^T$$

$$(1,2) := \hat{A}^T + \tilde{A} + \tilde{B}_2\hat{D}\tilde{C}_2$$

$$(1,3) := \tilde{B}_1W + \tilde{B}_2\hat{D}\tilde{D}_{21}W - (\tilde{C}_1X + \tilde{D}_{12}\hat{C})^T$$

$$(2,2) := \tilde{A}^TY + Y\tilde{A} + \hat{B}\tilde{C}_2 + \tilde{C}_2^T\hat{B}$$

$$(2,3) := Y\tilde{B}_1W + \hat{B}\tilde{D}_{21}W - \tilde{C}_1^T - \tilde{C}_2^T\hat{D}^T\tilde{D}_{12}^T$$

$$(3,3) := -\tilde{D}_{11}W - \tilde{D}_{12}\hat{D}\tilde{D}_{21}W - W^T\tilde{D}_{11}^T - W^T\tilde{D}_{21}^T\hat{D}^T\tilde{D}_{12}^T$$

where $W \in \mathbb{R}^{m\times m}$, $X \in \mathbb{R}^{n_P\times n_P}$, $Y \in \mathbb{R}^{n_P\times n_P}$, $\hat{A} \in \mathbb{R}^{n_P\times n_P}$, $\hat{B} \in \mathbb{R}^{n_P\times n_u}$, $\hat{C} \in \mathbb{R}^{n_y\times n_P}$, $\hat{D} \in \mathbb{R}^{n_y\times n_u}$, and the symbol $*$ is readily inferred by symmetry.

There has not yet an efficient algorithm for solving general BMI problem, instead, ILMI method is widely used for solving the BMI problems. In the following, we present two ILMI-based robust controller design methods on the basis of different ways of partitioning the BMI variables.

Method 1:

In this method, the BMI variables are partitioned into two groups, the group W and the group X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, and compute iteratively. The detailed synthesis procedure is outline in the following algorithm.

Algorithm 1:

Step 1. Suppose the generalized plant $P(s)$ is given; then the controller K can be computed via H_∞ theory. A lower bound of the robust stability margin γ can be estimated accordingly.

1.1 Compute an initial controller K where

$$K \overset{ss}{=} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$

via H_∞ theory with respect to the generalized plant P; then estimate the stability margin γ.

1.2 Compute

$$K_{eq} \overset{ss}{=} \begin{bmatrix} A_{K_{eq}} & B_{K_{eq}} \\ C_{K_{eq}} & D_{K_{eq}} \end{bmatrix}$$

$$:= \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}\left[I - \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}\right]^{-1}$$

1.3 Compute

$$\breve{P}_\gamma = \begin{pmatrix} \tilde{P}_{\gamma_{11}} & \tilde{P}_{\gamma_{12}} \\ \tilde{P}_{\gamma_{21}} & \tilde{P}_{\gamma_{22}} - \tilde{P}_{\gamma_{22}}(\infty) \end{pmatrix} = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & 0 \end{array}\right]$$

Step 2. Compute generalized stability multiplier W with increasing stability margin γ where the equivalent controller K_{eq} is fixed.

2.1 Increase γ; then compute $\tilde{M}_\gamma := F_l(\breve{P}_\gamma, K_{eq})$, afterward, solve the following equations,

$$W > 0; \quad (8)$$

$$\begin{pmatrix} A_{\tilde{M}_\gamma}Q + QA_{\tilde{M}_\gamma}^T & B_{\tilde{M}_\gamma}W - QC_{\tilde{M}_\gamma}^T \\ WB_{\tilde{M}_\gamma}^T - QC_{\tilde{M}_\gamma} & 2\varepsilon I - \left(D_{\tilde{M}_\gamma}W + WD_{\tilde{M}_\gamma}^T\right) \end{pmatrix} \le 0 \quad (9)$$

with $Q \in \mathbb{R}^{n_P+n_K}$ being the symmetric matrix and $\varepsilon > 0$.

2.2 Solve (8) and (9) for the stability multiplier, if feasible, back to step 2.1 till there is no significant increase in γ.

Step 3. Solve (6)-(7) for variables X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ with increasing stability margin γ where W are fixed.

3.1 Increase γ; then compute $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{D}_{11}$, $\tilde{D}_{12}$, $\tilde{D}_{21}$, $\tilde{D}_{22}$.

3.2 Solve (6) and (7) for variables X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, if feasible, back to step 3.1 till there is no significant increase in γ.

Step 4. Repeat step 2 and step 3, till the robustness is achieved or there is no significant increase in γ. Then, we can obtain a controller K_{eq}; afterward, the resulting controller is given by $K = K_{eq}\left(I + \tilde{D}_{22}K_{eq}\right)^{-1}$, in state space representation

$$K \overset{ss}{=} \begin{bmatrix} A_{K_{eq}} & B_{K_{eq}} \\ C_{K_{eq}} & D_{K_{eq}} \end{bmatrix}\left[I + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\begin{bmatrix} A_{K_{eq}} & B_{K_{eq}} \\ C_{K_{eq}} & D_{K_{eq}} \end{bmatrix}\right]^{-1}$$

with

$$D_{K_{eq}} = \hat{D};$$
$$C_{K_{eq}} = \left(\hat{C} - \hat{D}\tilde{C}_2 X\right) M^{-T};$$
$$B_{K_{eq}} = N^{-1}\left(\hat{B} - Y\tilde{B}_2 D_{K_{eq}}\right);$$
$$A_{K_{eq}} = N^{-1}\begin{bmatrix} \hat{A} - NB_{K_{eq}}\tilde{C}_2 X - Y\tilde{B}_2 C_{K_{eq}} M^T - \\ Y\left(\tilde{A} + \tilde{B}_2 D_{K_{eq}}\tilde{C}_2\right) X \end{bmatrix} M^{-T}.$$

where N and M so that $MN^T = I - XY$ (e.g., $N = I$, $M = I - XY$).

Method 2:

In this method, the BMI variables are partitioned into two groups, the group W, X, $\hat{A}, \hat{C}$ and the group X, Y, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, and compute iteratively. The synthesis procedure is similar to Algorithm 1. .

Step 4. Solve (5)-(7) for variables W, X, $\hat{A}, \hat{C}$ with increasing stability margin γ where $\hat{B}$, Y, $\hat{D}$ are fixed.

4.1 Increase γ; then compute $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{D}_{11}$, $\tilde{D}_{12}$, $\tilde{D}_{21}$, $\tilde{D}_{22}$.

4.2 Solve (5) and (7) for variables W, X, $\hat{A}, \hat{C}$, if feasible, back to step 3.1 till there is no significant increase in γ.

Step 5. Repeat Step 3 and Step 4, till the robustness is achieved or there is no significant increase in γ.

The resulting controller can be computed by the formula described in Step 4 of Algorithm 1.

Remark 1. There exist several other iterative schemes for solving the robust controller design problem in the derived BMI formulation as long as each of the following two groups of variables, (Y, $\hat{B}$, $\hat{D}$) and W, is solved in different phase. Note that the variations of the proposed methods stem from that the variables, X, $\hat{A}$, $\hat{C}$, can be either set free to solve or kept fixed in each individual phase. It is noted that the methods derived from the concept we pointed includes method 1 as a special case, whcih is simply the counterpart of the well known $D-K$ iteration.

Remark 2. For some practical problems, e.g., the flexible mechanical system example presented in [8], the generalized plants have the special property that $D_{21} = 0$. As can be checked, this additional information makes the term $\tilde{D}_{21}$ null too, which in turn eliminates several nonlinear coupled terms in (6) and (7). Thus the only constraint for applying ILMI method to solve the BMIs (6),(7) is to solve the variables Y and W in different phases. Moreover, each of the rest of the variables X, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ can be either treated as a variable to solve or kept fixed as a constant in each phase. With these observations, several iterative (synthesis) schemes like method 1 are possible for this class of special cases.

Remark 3. An equivalent condition to Theorem 1 (ii) is that $F_l(\hat{P}_\gamma, K)(s)$ is strictly positive real, where $\hat{P}_\gamma(s) = diag\left(W, I\right)\tilde{P}_\gamma(s)$. With the same derivation, this lead to another BMI formulation for the robust controller synthesis problem. Specifically, some terms are replaced as follows:

$$(1,3) = \tilde{B}_1 + \tilde{B}_2\hat{D}\tilde{D}_{21} - (W\tilde{C}_1 X + W\tilde{D}_{12}\hat{C})^T;$$
$$(2,3) = Y\tilde{B}_1 + \hat{B}\tilde{D}_{21} - \tilde{C}_1^T W^T - \tilde{C}_2^T \hat{D}^T \tilde{D}_{12}^T W^T;$$
$$(3,3) = -W\tilde{D}_{11} - W\tilde{D}_{12}\hat{D}\tilde{D}_{21} - \tilde{D}_{11}^T W^T - \tilde{D}_{21}^T \hat{D}^T \tilde{D}_{12}^T W^T.$$

Again, there exist several other iterative schemes for solving the robust controller design problem in the BMI formulation as long as each of the following two groups of variables, (X, $\hat{C}$, $\hat{D}$) and W, is solved in different phase. Note that the variations of the proposed methods stem from that the variables, Y, $\hat{A}$, $\hat{B}$, can be either set free to solve or kept fixed in each individual phase. Similrly, for the special case $D_{12} = 0$, the only constraint for applying ILMI method to solve the BMIs (6),(7) with the necessary replacement described above is to solve the variables X and W in different phases.

Remark 4. One of the advantages brought by the LMI approach developed by Scherer et al [6] is that strict properness of the resulting controller can be guaranteed if it is required. This can be done simply by setting $\hat{D} = 0$ (thus $K_{eq}(\infty) = 0$, which in turn implies that $K(\infty) = 0$) in (6) and (7).

Remark 5. While method 1 is the counterpart of the well known $D-K$ iteration which alwayes computes the multiplier and the controller in different phase, method 2 provides a mechanism for the constant multiplier and part of the controller parameters (i.e., X, $\hat{A}, \hat{C}$) to trade off in a single phase. This results in different controllers which may provide better performance.

4 Numerical Example

Consider a perturbed negative feedback system as depicted in Fig. 4,

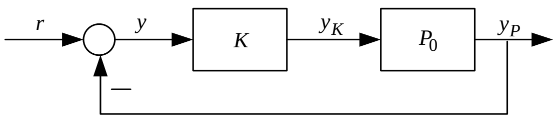

Fig. 4. perturbed negative feedback system

where the perturbed plant P_0 is a second-order linear time-invariant system with nonlinear time-varying uncertainties $\delta_i, i=1,2$, which has the following state-space representation.

$$\begin{cases} \dot{x}_P = \begin{bmatrix} -3 & 3 \\ 1 & 0 \end{bmatrix} x_P + \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta_2 \end{bmatrix} \right) y_K \\ y_P = \left(\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \Delta_1 \end{bmatrix} \right) x_P \end{cases}$$

In terms of the standard $\Delta - P - K$ framework where $\Delta := diag(\delta_1, \delta_2)$, the generalized plant is given by

$$P(s) \leftrightarrow \left[\begin{array}{cc|cc|c} -3 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline -1 & -3 & -1 & 0 & 0 \end{array} \right]$$

According to $P(s)$, we can obtain the numerically computational data (i.e. stability margin γ, generalized stability multiplier W and controller K) in Table 2 by applying H_∞ control theory, method 1 and method 2.

Table 2

H_∞ control theory	
γ	0.1839
W	None
K	$\frac{7335.2s + 27802}{s^2 + 1367.6s + 2142.7}$
Method 1	
γ	0.2148
W	$\begin{bmatrix} 0.38329 & 0 \\ 0 & 6.5206 \end{bmatrix}$
K	$\frac{4.24s^2 + 6403.4s + 24103}{s^2 + 437.56s - 1235.7}$
Method 2	
γ	0.2278
W	$\begin{bmatrix} 0.022825 & 0 \\ 0 & 0.32629 \end{bmatrix}$
K	$\frac{-1.8569 \times 10^{-4} s^2 + 7.1299 \times 10^5 s + 2.6877 \times 10^6}{s^2 + 43129s - 1.6325 \times 10^5}$

Compared to the result of H_∞ control theory, it shows that method 1 and method 2 yield a much better value in γ.

5 Conclusion

In this work, a new BMI formulation was established for the robust controller design for the systems with bounded nonlinear time-varying uncertainties. Different synthesis schemes were proposed. Specifically, the BMI variables were divided into two groups and solved iteratively over LMIs in separate phases. Particularly, it is noted that the counterpart of the well known $D-K$ iteration is a special case of the methods we proposed. In addition, the advantage of applying the work by Scherer et al was revealed. The given numerical example showed the effectiveness of the proposed methods.

Acknowledgement

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