Robust Controller Designs for Systems with Real Parametric Uncertainties

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Abstract: - In this work, we revisit the robust stability margin optimization problem. Two robust controller designs are proposed for systems subject to real parametric uncertainty, which employ the recently developed methods for computing the real μ upper bounds and the linear matrix inequality (LMI) method developed by Scherer et al for computing strictly positive real (SPR) controllers. Comparisons between the two proposed methods as well as the advantage of utilizing Scherer et al's method are discussed.

Key-Words: - real parametric uncertainty, robust controller synthesis, stability multiplier, LMI

1 Introduction

In the last two decades, μ analysis [1,2,3,4,5,6] and synthesis [7,8,9,10,11] have emerged to be powerful tools for analyzing and synthesizing robust controllers for systems subject to multiple sources of parametric and/or dynamic uncertainties. In 1985 the well known D-K iteration [7] was proposed to produce robust controllers for the systems with structured dynamic uncertainties, which is essentially based on iterating between the phase of computing the complex μ upper bound (or the optimal multipliers) with the controller fixed, and the phase of H_∞ optimization with the multiplier fixed. Thereafter, almost all of the currently existing μ synthesis methods follow this two-phase iterative scheme.

In early research about the phase of computing the optimal multipliers, the scalings are available via solving a set of LMIs at several grid frequencies, and curve fitting is performed to obtain a finite-dimensional transfer function representation of them. The quality of the curve fits for the scalings is not easy to control (especially the matrix-valued multipliers satisfying certain properties) and it often results in excessively high-order controllers in the subsequent stage if the approximation error is intended to be made small. To alleviate this difficulty, Ly et al [3] consider multipliers of the particular form

$$W(s) := \sum_{i=1}^{l} \Theta_i \frac{s^i}{d(-s)d(s)}$$

where Θ_i are matrix variables and $d(s)$ is a fixed order polynomial with no zeros on the imaginary axis. A similar method employing rational functions as a basis was proposed in [4] at about the same time. While curve fitting of the scalings is no longer required and good estimates of the μ upper bound could be rendered by both of the methods, both of them rely heavily on a proper choice of the bases, equivalently, the choice of the poles of the multipliers. Later, in [5] a skillful LMI method was proposed for computing the optimal multipliers. Particularly, there is no need to choose the poles of the multipliers a priori, yet a lower bound constraint on the multiplier order (at least as large as the order of closed-loop system) is assumed for the approach. This generally improves the computation of the real μ upper bound at the price of employing high-order multipliers.

Nowadays the computation of the full-order H_∞ and SPR controller is well understood (e.g., two Riccati approach [12] or LMI approach [13,14]) and different ways of computing the optimal multipliers are available; however, study of integrating the recently developed methods [4,5,14] and making comparisons is rare. Therefore, it is our purpose to present different robust controller designs via integrating the methods and thus propose new ones.

The rest of the paper is organized as follows. Section 2 gives the formal problem statement and some preliminaries for future developments. Section 3 presents the two proposed approaches for the robust controller synthesis problem. Section 5 is the conclusions.

2 Problem Formulation

Notation

In this section some notations and background material will be first presented. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. Given a matrix A, A^* means the complex conjugate transpose of A. Let H_∞ be the subspace of analytic and bounded transfer functions in the open right-half plane and RH_∞ denotes the space of all proper and real-rational transfer functions in H_∞. The H_∞ norm of a stable transfer function $G(s)$ is defined as: $\|G\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}[G(s)] = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[G(j\omega)]$, and $G(s)$ is called unimodular in H_∞ if $G^{-1}(s) \in H_\infty$. Let U(RH_∞) denote the set of all unimodular transfer matrices in RH_∞. Let RF be the set of real-rational, proper transfer functions. Let $X^\sim(s) := X^T(-s)$. A square matrix transfer function $X(s)$ is said strictly positive real (SPR) if (i) $X(s)$ is analytic in the closed right-half complex plane, and (ii) $(1/\varepsilon)I > \frac{1}{2}\left(X(j\omega) + X^*(j\omega)\right) > \varepsilon I$ for some $\varepsilon > 0$ and all ω on the extended real line [13]. Let $F_l(\bullet,\bullet)$ denote the lower linear fraction representation, see [14]. The symbol $S_R(X)$ is defined as:

$$S_R(X) = \left\{block - diag(S_1,\dots,S_L) : S_i \in X^{m_i\times m_i}, i = 1,\dots,L\right\}$$

where $X = \mathbb{R}$ or $\mathbb{C}$ or RF. The state-space realization of $H(s)$ is denoted as:

$$H(s) \underset{=}{ss} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Problem Description

The popular $\Delta - P - K$ robust controller synthesis paradigm is considered. Throughout this paper, we assume the uncertainty Δ belongs to the parametric structured uncertainty set defined as follows:

$$\Delta_r := \left\{block - diag(\delta_1 I_{m_1},\dots,\delta_L I_{m_L}) : \delta_i \in R, i = 1,\dots,L\right\}$$

The symbol P denotes the generalized plant, including the nominal plant, weighting functions, etc, described by

$$P \quad \left\{ \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \right. \qquad (1)$$

where $x \in \mathbb{R}^{n_P}$, $w \in \mathbb{R}^m$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^m$, and $y \in \mathbb{R}^{n_y}$ with $m := m_1 + \cdots + m_L$. The symbol K denotes a dynamic controller of the form

$$K \quad \left\{ \begin{aligned} \dot{x}_K &= A_K x_K + B_K y \\ \mathrm{u} &= \mathrm{C}_K x_K + D_K y \end{aligned} \right. \qquad (2)$$

where $x \in \mathbb{R}^{n_K}$, to be designed. Our goal is to find a dynamic controller K of form (2) to enlarge the robust stability margin, defined as the largest size of the structured uncertainties against which the nominal system $M := F_l(P,K)$ is robustly stable. Specifically, for all $\Delta \in \Delta_r$ with $\|\Delta\| \le \gamma$ (i.e., $|\delta_i| \le \gamma$ for all i) we want to design a dynamic controller of form (2) to maximize the value γ such that the nominal system (1)-(2) is robustly stable. Obviously, application of the H_∞ control theory immediately yields a lower bound of the robust stability margin being the value $1/\|M\|_\infty$. But, this is usually an overly conservative result because the information of the uncertainty, that the uncertainty is real parametric and structured, was not exploited during the design. To reduce the conservatism, an improved technique which employs passivity theorem with multipliers has been addressed in [2,4,6,8,10], as shown in Fig. 1, where $\tilde{\Delta} = (I + \gamma^{-1}\Delta)(I - \gamma^{-1}\Delta)^{-1}$, $\tilde{M}_\gamma := (I - \gamma M)(I + \gamma M)^{-1} = F_l\left(\tilde{P}_\gamma, K\right)$.

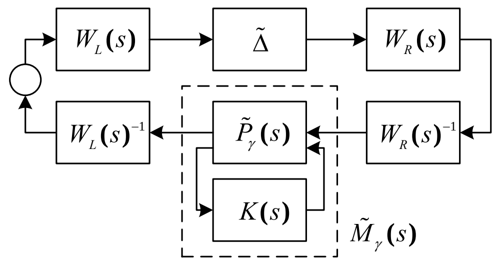

Fig. 1 Passivity framework with multipliers

It is easy to check that for any $\Delta \in \Delta_r$ with $\|\Delta\| \le \gamma$, $\tilde{\Delta}$ is a constant diagonal matrix with entries in $[0,\infty)$. A well known sufficient condition for the robust synthesis problem is thus to maximize the value γ via finding a dynamic controller K of form (2) and suitable stability multipliers W_L and W_R in the set $S_R(RF)$, satisfying (i) W_L, W_R, W_L^{-1} and W_R^{-1} are in RH_∞, (ii) $W_L W_R$ is strictly positive real (SPR), and (iii) $W_L^{-1}\tilde{M}_\gamma W_R^{-1}$ is SPR [2,4,6,8,10]. By letting $W = W_R^{-1}W_L^\sim$ (possibly non-causal), it has been shown that the multiplier conditions described above are equivalent to the following frequency domain conditions which must hold for some $\varepsilon > 0$ and for all $\forall \omega \in \mathbb{R} \cup \{\infty\}$,

$$\tilde{M}_\gamma(j\omega)W(j\omega) + W(j\omega)^* \tilde{M}_\gamma(j\omega)^* > 2\varepsilon I , \quad (3)$$

$$W(j\omega) + W(j\omega)^* > 0 . \quad (4)$$

In summary, the robust synthesis problem we consider is to maximize the value γ via finding a dynamic controller K of form (2) and a generalized stability multiplier $W \in S_R(RF)$ such that conditions (3) and (4) hold.

For the latter development, it is required to know the state-space realization of the sector transformed plant $\tilde{P}_\gamma = S \star \Gamma P$, where

$$S = \begin{pmatrix} I_m & -\sqrt{2}I_m \\ \sqrt{2}I_m & -I_m \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma I_m & 0 \\ 0 & I_{n_y} \end{pmatrix}$$

and the symbol $\star$ means star product [12]. Furthermore, it is assumed that $\tilde{P}_\gamma$ has a state-space realization as follows.

$$\tilde{P}_\gamma(s) \underset{=}{ss} \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right]$$

3 Robust Controller Design

In this section we present two closely related approaches for the robust controller synthesis problem described in Section 2. The two approaches mimic the well known D-K iteration for complex μ synthesis [7], which essentially involve iterations between computing the real μ upper bound and solving a SPR control problem, with the only difference lying on utilizing different methods [4,5] for computing the real μ upper bound.

Method 1:

The design procedure of method 1 involves iterations between computing the real μ upper bound (analysis phase) and solving for a SPR controller (synthesis phase). For comparison purpose, a brief review of the methods used in the two phases is presented.

In the analysis phase, i.e., computing the real μ upper bound (equivalently, find a generalized stability multiplier W maximizing γ) with controller fixed, the method developed by [4] is used. It is assumed that the generalized stability multiplier W is an affine parameterization of certain fixed, real-rational, block-diagonal transfer matrices $M_i \in S_R(RF)$, $i = 0,\ldots,l$, i.e.,

$$W(s) := M_0(s) + \sum_{i=1}^{l} \theta_i M_i(s)$$

Thus W has a canonical form realization as

$$W(s) \underset{=}{ss} \begin{bmatrix} A_W & B_W(\theta) \\ C_W & D_W(\theta) \end{bmatrix},$$

with $A_W \in \mathbb{R}^{n_W \times n_W}$, $C_W(\theta) \in \mathbb{R}^{m \times n_W}$, and $B_W(\theta) \in \mathbb{R}^{n_W \times m}$, $D_W(\theta) \in \mathbb{R}^{m \times m}$ depending affinely on the real parameter vector $\theta = [\theta_1, \cdots, \theta_l]$. Further, assuming that $\tilde{M}_\gamma$ has the following system realization $\left(A_{\tilde{M}_\gamma}, B_{\tilde{M}_\gamma}, C_{\tilde{M}_\gamma}, D_{\tilde{M}_\gamma}\right)$, it is easy to verify that the cascaded system $\tilde{M}_\gamma W$ has a state-space realization $\left(A_{\tilde{M}_\gamma W}, B_{\tilde{M}_\gamma W}(\theta), C_{\tilde{M}_\gamma W}, D_{\tilde{M}_\gamma W}(\theta)\right)$ where

$$A_{\tilde{M}_r W} = \begin{bmatrix} A_W & 0 \\ B_{\tilde{M}_r} C_W & A_{\tilde{M}_r} \end{bmatrix}; \; B_{\tilde{M}_r W}(\theta) = \begin{bmatrix} B_W(\theta) \\ B_{\tilde{M}_r} D_W(\theta) \end{bmatrix};$$

$$C_{\tilde{M}_r W} = [D_{\tilde{M}_r} C_W \quad C_{\tilde{M}_r}]; \; D_{\tilde{M}_r W}(\theta) = [D_{\tilde{M}_r} D_W(\theta)] .$$

Note that only $B_{\tilde{M}_r W}(\theta)$ and $D_{\tilde{M}_r W}(\theta)$ depend affinely on θ. It follows from the robust stability conditions (3)-(4) and the generalized positive real lemma that the nominal system $F_l(P,K)$ described by (1)-(2) is robustly stable against the set of real parametric uncertainties $\Delta \in \Delta_r$, with size no greater than γ, if there exist matrices $Q_W = Q_W^T \in \mathbb{R}^{n_W \times n_W}$, $Q = Q^T \in \mathbb{R}^{(n_{\tilde{M}_\gamma} + n_W) \times (n_{\tilde{M}_\gamma} + n_W)}$, real parameter vector θ and $\varepsilon > 0$, such that the following LMIs hold:

$$\begin{bmatrix} A_W Q_W + Q_W A_W^T & B_W(\theta) - Q_W C_W^T \\ B_W(\theta)^T - C_W Q_W & 2\varepsilon I - (D_W(\theta) + D_W(\theta)^T) \end{bmatrix} \le 0, \quad (5)$$

$$\begin{bmatrix} A_{\tilde{M}_r W} Q + Q A_{\tilde{M}_r W}^T & B_{\tilde{M}_r W}(\theta) - Q C_{\tilde{M}_r W}^T \\ B_{\tilde{M}_r W}(\theta)^T - C_{\tilde{M}_r W} Q & 2\varepsilon I - (D_{\tilde{M}_r W}(\theta) + D_{\tilde{M}_r W}(\theta)^T) \end{bmatrix} \le 0 . \quad (6)$$

Notice that the value γ, a lower bound of the robust stability margin, assessed via the preceding robustness test relies heavily on the obtained generalized stability multiplier W. In particular, a proper choice of its poles (i.e., the poles of the fixed transfer matrices M_i) matters much.

In the synthesis phase the generalized stability multiplier is fixed. In view of Fig. 1, we need to find a controller such that the transfer matrix $W_L^{-1} F_l(\tilde{P}_\gamma, K) W_R^{-1}$ is SPR. We attempt to use the recently developed LMI synthesis method [14]. To proceed, spectral factorization of the generalized stability multiplier W (obtained in the analysis phase) as $W = W_R^{-1} W_L^{\sim}$, where $W_R, W_L \in \mathrm{U}(RH_\infty)$, is first carried out. Then an augmented plant denoted by $\tilde{P}_W$ is formed via incorporating W_L^{-1} and W_R^{-1} into $\tilde{P}_\gamma$, i.e.,

$$\tilde{P}_W := \begin{pmatrix} W_L^{-1} & 0 \\ 0 & I_{n_y} \end{pmatrix} \tilde{P}_\gamma \begin{pmatrix} W_R^{-1} & 0 \\ 0 & I_{n_u} \end{pmatrix} =: \begin{pmatrix} \tilde{P}_{W_{11}} & \tilde{P}_{W_{12}} \\ \tilde{P}_{W_{21}} & \tilde{P}_{W_{22}} \end{pmatrix}.$$

However, since $\tilde{P}_{W_{22}}(\infty) = \tilde{D}_{22}$ usually is not null, direct application of the method [14] to compute a controller is prohibited. To circumvent this difficulty, loop transform technique is introduced, as is illustrated in Fig. 2.

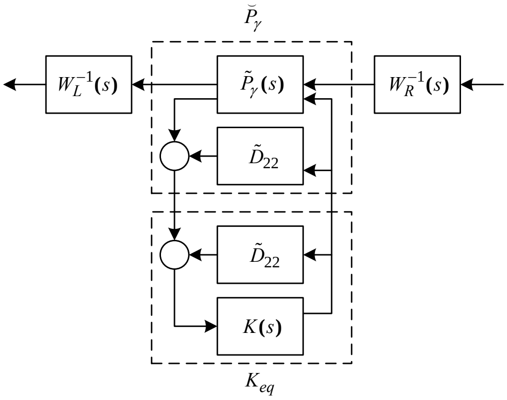

Fig. 2 Equivalent system via loop transformation

Specifically, the transformed plant $\breve{P}_\gamma$ and the transformed controller K_{eq} are defined as follows:

$$\breve{P}_\gamma := \begin{pmatrix} \tilde{P}_{\gamma_{11}} & \tilde{P}_{\gamma_{12}} \\ \tilde{P}_{\gamma_{21}} & \tilde{P}_{\gamma_{22}} - \tilde{D}_{22} \end{pmatrix} \underline{\underline{ss}} \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & 0 \end{array} \right] \tag{7}$$

and

$$K_{eq} := K\left(I - \tilde{D}_{22}K\right)^{-1} \tag{8}$$

Next, the new interconnection $\breve{P}_W$ is formed as follows:

$$\breve{P}_W := \begin{pmatrix} W_L^{-1} & 0 \\ 0 & I_{n_y} \end{pmatrix} \breve{P}_\gamma \begin{pmatrix} W_R^{-1} & 0 \\ 0 & I_{n_u} \end{pmatrix} \underline{\underline{ss}} \left[\begin{array}{c|cc} \breve{A} & \breve{B}_1 & \breve{B}_2 \\ \hline \breve{C}_1 & \breve{D}_{11} & \breve{D}_{12} \\ \breve{C}_2 & \breve{D}_{21} & \breve{D}_{22} \end{array} \right].$$

where $\breve{A} \in \mathbb{R}^{(n_P+n_W)\times(n_P+n_W)}$, $\breve{D}_{11} \in \mathbb{R}^{m\times m}$ and $\breve{D}_{22} = 0^{n_y \times n_u}$. Now it's ready to apply the method in [14]. By [14], there exists a K_{eq} such that $F_l(\breve{P}_W, K_{eq})$ (equivalently $W_L^{-1}F_l(\tilde{P}_\gamma, K)W_R^{-1}$) is SPR if and only if there exist matrices $X = X^T \in \mathbb{R}^{(n_P+n_W)\times(n_P+n_W)}$, $Y = Y^T \in \mathbb{R}^{(n_P+n_W)\times(n_P+n_W)}$, $\hat{A} \in \mathbb{R}^{(n_P+n_W)\times(n_P+n_W)}$, $\hat{B} \in \mathbb{R}^{(n_P+n_W)\times n_y}$, $\hat{C} \in \mathbb{R}^{n_u \times (n_P+n_W)}$ and $\hat{D} \in \mathbb{R}^{n_u \times n_y}$ such that the following LMIs are feasible.

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{9}$$

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^T & H_{22} & H_{23} \\ H_{13}^T & H_{23}^T & H_{33} \end{bmatrix} < 0 \tag{10}$$

where

$$H_{11} := \breve{A}X + X\breve{A}^T + \breve{B}_2\hat{C} + (\breve{B}_2\hat{C})^T$$

$$H_{12} := \hat{A}^T + (\breve{A} + \breve{B}_2\hat{D}\breve{C}_2)$$

$$H_{13} := (\breve{B}_1 + \breve{B}_2\hat{D}\breve{D}_{21}) - (\breve{C}_1X + \breve{D}_{12}\hat{C})^T$$

$$H_{22} := \breve{A}^TY + Y\breve{A} + \hat{B}\breve{C}_2 + (\hat{B}\breve{C}_2)^T$$

$$H_{23} := (Y\breve{B}_1 + \hat{B}\breve{D}_{21}) - (\breve{C}_1 + \breve{D}_{12}\hat{D}\breve{C}_2)^T$$

$$H_{33} := -(\breve{D}_{11} + \breve{D}_{12}\hat{D}\breve{D}_{21}) - (\breve{D}_{11} + \breve{D}_{12}\hat{D}\breve{D}_{21})^T$$

In the affirmative case, one can choose N and M so that $MN^T = I - XY$ (e.g., $N = I$, $M = I - XY$). Then a dynamic controller K which achieves the goal is given by $K = K_{eq}\left(I + \tilde{D}_{22}K_{eq}\right)^{-1}$, in state space representation

$$K \underline{\underline{ss}} \begin{bmatrix} A_{K_{eq}} & B_{K_{eq}} \\ C_{K_{eq}} & D_{K_{eq}} \end{bmatrix} \left[I + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} A_{K_{eq}} & B_{K_{eq}} \\ C_{K_{eq}} & D_{K_{eq}} \end{bmatrix} \right]^{-1}$$

with

$$D_{K_{eq}} = \hat{D};$$

$$C_{K_{eq}} = \left(\hat{C} - \hat{D}\breve{C}_2X\right)M^{-T};$$

$$B_{K_{eq}} = N^{-1}\left(\hat{B} - Y\breve{B}_2D_{K_{eq}}\right);$$

$$A_{K_{eq}} = N^{-1} \begin{bmatrix} \hat{A} - NB_{K_{eq}}\breve{C}_2X - Y\breve{B}_2C_{K_{eq}}M^T - \\ Y\left(\breve{A} + \breve{B}_2D_{K_{eq}}\breve{C}_2\right)X \end{bmatrix} M^{-T}.$$

A detailed synthesis procedure integrating the two phases outlined above is presented in the following D-K iteration like algorithm.

Algorithm 1:

1. Compute a suboptimal H_∞ controller K such that the value γ which satisfies $\|\gamma F(P,K)\|_\infty < 1$ is maximized. Then compute $K_{eq} = K\left(I - \tilde{D}_{22}K\right)^{-1}$.
2. Find a generalized stability multiplier W maximizing γ by solving (5) and (6) where the transformed controller K_{eq} (by eq.(8)) is kept fixed. The details are as follows:
   2.1 Increase γ. Compute $\breve{P}_\gamma$ ((by eq.(7))) and $\tilde{M}_\gamma := F_l(\breve{P}_\gamma, K_{eq})$. Select the poles for W

(equivalently, set the values of A_W and C_W for the canonical representation of W).

2.2 Solve (5) and (6) for the variables Q_W, Q, θ and ε. Back to step 2.1 till there is no significant increase in γ.

2.3 Compute the generalized stability multiplier $W(s) = C_W(sI - A_W)^{-1}B_W(\theta) + D_W(\theta)$.

3. Find a transformed controller K_{eq} with the generalized stability multiplier W fixed. The details are as follows:

3.1 Let the generalized stability multiplier W be factored as $W = W_R^{-1}(W_L)^{\sim}$, where $W_R, W_L \in \mathrm{U}(RH_\infty)$.

3.2 Compute the interconnection $\breve{P}_W = \begin{pmatrix} W_L^{-1} & 0 \\ 0 & I \end{pmatrix} \breve{P}_\gamma \begin{pmatrix} W_R^{-1} & 0 \\ 0 & I \end{pmatrix}$.

3.3 Solve (9) and (10) for K_{eq} such that $F_l(\breve{P}_W, K_{eq})$ is SPR.

4. Repeat step 2 and step 3, till there is no significant increase in γ.
5. The resulting controller is given by

$$K = K_{eq}\left(I + \tilde{D}_{22}K_{eq}\right)^{-1}.$$

Remark 1. Algorithm 1 successfully combines the method of [4] for searching suitable multipliers with the method [14] for computing a SPR controller to yield a robustly stabilizing controller. The resulting controller is in general proper but not strictly proper. Algorithm 1 is quite flexible regarding this point. That is, if the strict properness property is desired, one can simply set $\hat{D} = 0$ (thus $K_{eq}(\infty) = 0$, which in turn implies that $K(\infty) = 0$) in (8) without altering the convex property of the solutions of the matrix inequality.

In the analysis phase of Algorithm 1, the method [4] is used. It is noted that the poles of the generalized stability multiplier must be selected a priori so as to make the multiplier searching problem convex and easily solved. While good choices can be made by experienced experts, there is no general guideline on how to choose the poles. An approach to alleviate this problem was addressed in [5]. With this replacement, we obtain an alternative D-K iteration like robust controller synthesis method as follows.

Method 2:
The design is similar to Method 1. The only difference is to replace the analysis phase (step 2) of Algorithm 1 with the method proposed in [5]. A brief review of the approach [5] is introduced as follows: Consider the stable system $\tilde{M}_\gamma := F_l(\tilde{P}_\gamma, K)$. As before, let $n_{\tilde{M}_\gamma}$ denote the order of the stable system $\tilde{M}_\gamma$ and n_W denote the order of generalized stability multiplier W to be computed. Under the constraint $n_W \geq n_{\tilde{M}_\gamma} = n_P + n_K$ where n_p and n_K denote the order of the generalized plant P and the controller K respectively, suppose that there exist a positive number ε, matrices $P = P^T \in \mathbb{R}^{(n_P+n_K)\times(n_P+n_K)}$, and block-diagonal matrices $X \in \mathbb{R}^{n_W \times n_W}$, $Y \in \mathbb{R}^{m \times n_W}$, $P_W = P_W^T \in \mathbb{R}^{n_W \times n_W}$, $B_W \in \mathbb{R}^{n_W \times m}$, $D_W \in \mathbb{R}^{m \times m}$, with P_W non-singular, such that the following LMIs hold

$$\begin{bmatrix} X^T + X & B_W - Y^T \\ B_W^T - Y & 2\varepsilon I - (D_W + D_W^T) \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{bmatrix} < 0 \tag{12}$$

where

$$L_{11} = \begin{bmatrix} X^T + X & Y^T B_{\tilde{M}_r}^T + P_W \Gamma_0 A_{\tilde{M}_r}^T + X\Gamma_0 \\ * & PA_{\tilde{M}_r}^T + A_{\tilde{M}_r}P + \Gamma_0^T Y^T B_{\tilde{M}_r}^T + B_{\tilde{M}_r} Y \Gamma_0 \end{bmatrix};$$

$$L_{12} = \begin{bmatrix} B_W - Y^T D_{\tilde{M}_r}^T - P_W \Gamma_0 C_{\tilde{M}_r}^T \\ B_{\tilde{M}_r} D_W - \Gamma_0^T Y^T D_{\tilde{M}_r}^T - PC_{\tilde{M}_r}^T \end{bmatrix};$$

$$L_{22} = 2\varepsilon I - D_{\tilde{M}_r} D_W - D_W^T D_{\tilde{M}_r}^T;$$

$$\Gamma_0 := \begin{bmatrix} I_{n_p+n_K} \\ 0 \end{bmatrix} \in R^{n_W \times (n_P+n_K)}.$$

Then $W(s) := Y(sP_W - X)^{-1}B_W + D_W$ is a generalized stability multiplier satisfying conditions (3) and (4).

The paper [5] gives a sufficient condition for searching for a generalized stability multiplier via solving a couple of LMIs (11)-(12). In comparison with [4], there is apparently no need to choose the poles for the generalized stability multiplier a priori. This is an advantage of this method. An alternative synthesis procedure is thus established by employing this method [5]. Algorithm 1 is revised accordingly. Specifically, step 2 of Algorithm 1 is modified as follows:

Find a generalized stability multiplier W maximizing γ by solving (11) and (12) where the transformed controller K_{eq} is kept fixed. The details are as follows:

2.1 Increase γ. Compute $\breve{P}_\gamma$ and $\tilde{M}_\gamma := F_l(\breve{P}_\gamma, K_{eq})$.

Set $\Gamma_0 := \begin{bmatrix} I_{n_p+n_K} \\ 0 \end{bmatrix} \in R^{n_W \times (n_P+n_K)}$.

2.2 Solve (11) and (12) for the LMI variables X, Y, P_W, B_W, D_W and ε. Back to step 2.1 till there is no significant increase in γ.

2.3 Compute the generalized stability multiplier $W(s) := Y(sP_W - X)^{-1} B_W + D_W$.

Remark 2. Despite the method [5] provides an advantage that there is no need to choose the poles for the generalized stability multiplier a priori, application of the method is restrictive with the constraint that $n_W \geq n_P + n_K$. Because of this constraint and that the method [14] always produces full-order controller (i.e., the controller is of the same order with the augmented plant $\breve{P}_W$), it is expected that the order of the resulting controllers will dramatically increase as the iteration number of the synthesis procedure goes up. Specifically, the order of the resulting controller at the i-th iteration is at least $(2i-1)n_p$, where an iteration is realized to be consisting of computing controller and multiplier once.

4 Conclusions

Two robust controller designs were proposed for systems subject to real parametric uncertainties. The methods mimic the well known D-K iteration procedure for complex μ synthesis by integrating the recently developed methods for computing the optimal generalized stability multipliers and the strictly positive real controller. No curve fitting is required during the design, which thus reduces the error incurred in this step. But as usual the two methods often produce controllers of high order.

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