Exact Variational Principles of 3-D Unseady Navier-Stokes Equations

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Abstract:-As is well known, the search for the exact variational principles (VP) for the full Navier-Stokes equations of 3-D viscous flow is an extremely difficult, still open problem in fluid mechanics. Just recently this problem for the steady flow case has been successfully solved for the first time via a systematic reversed deduction approach incorporating a method of undetermined function. As a further development of that, two VP families are generated herein for the unsteady flow case, providing a new rigorous theoretical basis for the finite element analysis of unsteady viscous flow, especially for the direct numerical simulation (DNS)of turbulent flow.

Key-Words:- Navier-Stokes equations, Variational principle, Viscous flow, Direct numerical simulation of turbulent flow, Finit element method

1 Introduction

The search for exact (genuine) variational principles (VP) for the full Navier-Stokes (N-S) equations of viscous flow, as is well-known, is an extremely difficult, longstanding open problem in fluid mechanics. Since the derivation of the N-S equations by Navier (1822) and Stokes (1845), up to now, in the literature there exist genuine VP only for two special cases: either for the inviscid flow {Bateman's VP (1929); Herivel-Lin's VP (1955-1959)} or for the slow viscous flow with negligible inertial effect {Helmholtz's minimum dissipation principle (1868)}[1-5]. Due to the important role played by the N-S equations in science and engineering, especially since the advent and the widespread application of the finite element method in the mid 20th century, a great deal of research interest and effort has been dedicated to the search for VP of the N-S equations[1-13,], unfortunately, however, only limited progress has been achieved. Despite the conclusions about the nonexistence of VP for the N-S equations reached by Milikan[6] and Finlayson[1], a variety of non-standard approximate VP models such as the restricted VP, quasi-VP and pseudo-VP and so on has been proposed[1,7,8,]. For the full N-S equations in primitive variables, velocity $\vec{w}(=u_i\vec{e}_i$, $\vec{e}_i$—unit vector along the coordinate x_i, i=1, 2, 3) and pressure p, Milikan, as early as in 1929 [6], first gave a negative answer to the question of whether there exists a VP, assuming that the lagrangian was a polynomial of the velocities and their derivatives. Finlayson proved in 1972[1] by means of the Vainberg's theorem that the $\vec{w}$-P formulation of the N-S equations had a VP only if the differential operator was a potential one, i.e. either $\vec{w}\times(\nabla\times\vec{w})=0$ or $\vec{w}\cdot\nabla\vec{w}=0$ (i.e. Stokes flow). On the other hand, Carey was the first to notice that the potentiality of the operator is the sufficient, but not necessary, condition for the existence of a VP[10]. Tonti even claimed that VP can always be formulated for every differential equation by an integrating operator[11], however, its application to flow problems, especially to N-S equations, is still associated with formidable difficulties. Another possible way out is to define an enlarged(composite) system and to derive for it a composite VP (e.g. by using Lagrange multipliers)[1, 3, 12] whose Euler equations consist of not only the N-S equations but also the adjoint equations. Apparently, this approach suffers from the serious shortcoming that the number of the unknown functions involved is doubled so that the problem is significantly complicated and requires much more mathematical and computational work and is time-consuming and costly. In Ref.[13] Ecer employed this method along with a Clebsch-like transformation, but he did not succeeded in reducing the number of unknowns to the original one.

To bypass these difficulties, it should be especially pointed out that although the existence of primary VP{i.e.VP in terms of the primary (primitive) variables $\vec{w}$, p} is ruled out by the Vainberg's theorem it would be, nevertheless, still possible instead to establish VP of alternative types: mixed VP {i.e.VP in terms of some primary variables and some adjoint(dual)variables} and/or dual VP(i.e.VP in terms of dual variables)[19]. Such a primary/dual program approach is quite common in linear and nonlinear programming theory, where the dual program is preferred to use if it is simpler to solve than the primary one. Proceeding just in this way, in a previous paper[15] the problem of generating genuine (exact) VP for the N-S equations in 3-D steady flow was successfully solved for the first time by means of a systematic approach to the search and transformation for VP and generalized VP suggested previously by the present author[16, 17]. The essence of this systematic approach consists in the properly joined use of a new method of undetermined function and a reversed deduction method via Lagrange multipliers.

In the present paper a similar approach[15] is followed to extend the mixed and dual VP of Ref. [15] to the unsteady 3-D N-S equations.

2 Derivation of the First VP Family

2.1 Navier-Stokes Equations for 3-D Unsteady Incompressible Viscous Flow

Using the Einstein's index summation convention, these equations for the Newtonian fluid have the following nondimensional form[18]

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (1')$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} - \nu \nabla^2 u_i - f_i = 0 \qquad (2)$$

where $P = p/\rho$; $\vec{f}$ ——body force per unit mass; i, j=1~3; ν =1/Re.

For the convenience of the later derivation of the VP, it is expedient to introduce the pseudo-stream functions Π_1 and Π_2 via the following relations[14]

$$\frac{\partial \Pi_1}{\partial x_3} = u_1 \ , \quad \frac{\partial \Pi_2}{\partial x_3} = u_2 \ , \quad \frac{\partial \Pi_1}{\partial x_1} + \frac{\partial \Pi_2}{\partial x_2} = -u_3$$

(1)

It is easy to see that the continuity Eq.(1') is identically satisfied by Eq.(1), so that later on we will always use Eq.(1) instead of Eq.(1').

2.2 Three Mixed VP

To derive the VP for the N-S equations in the form of Eqs.(1)&(2),according to the systematic approach suggested in Refs.[16, 17] a trial functional *I* should be constructed first via the undetermined function $F(u_i, P, \Pi_1, \Pi_2)$ and the Lagrange multipliers λ_k (k=1~6) as follows:

$$\begin{aligned} I(u_i, P, \Pi_1, \Pi_2, \lambda_k) = \int_t \iiint_{(\Omega)} \{ & F(u_i, P, \Pi_1, \Pi_2) \\ & + \lambda_i (\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} - \nu \nabla^2 u_i - f_i) \\ & + \lambda_4 (u_1 - \frac{\partial \Pi_1}{\partial x_3}) + \lambda_5 (u_2 - \frac{\partial \Pi_2}{\partial x_3}) \\ & + \lambda_6 (u_3 + \frac{\partial \Pi_1}{\partial x_1} + \frac{\partial \Pi_2}{\partial x_2}) \} \mathrm{dVdt} \end{aligned} \qquad (3)$$

with $dV=dx_1dx_2dx_3$.

At this point it should be emphasized that in order to convert the N-S Eqs.(1) & (2) to some part of the Euler equations of the functional *I*, it is necessary and sufficient to take the undetermined function *F* free of the multipliers λ_k. Then by changing the form of the function *F* we can change accordingly the form of the adjoint equations, Eqs. (4A-4D). Thus, the incorporation of the undetermined function *F* in the functional *I* offers us a new freedom to facilitate and simplify the further treatment of the problem and just this freedom can be used to advantage in alternative ways, for example, for considerably extending the applicability range of the Lagrange multiplier method in deriving and transforming VP or for removing the variational crisis[16, 17] and so on.

For the present problem under consideration the next key step is to try to identify F and λ_k in such a way that the total number of unknowns in Eq.(3) (12 variables: $u_i, P, \Pi_1, \Pi_2, \lambda_1 \sim \lambda_6$) could be reduced to 6 {equal to the number of unknowns in the original Eqs. (1) & (2)} by eliminating any 6 of them. For this purpose, we set $\delta I = 0$, leading to the following set of the Euler equations (i, j=1~3; k=1~6):

$\delta\lambda_k$: Eqs. (1) & (2)

$$\delta u_i: \frac{\delta F}{\delta u_i} - \frac{\partial \lambda_i}{\partial t} - \frac{\partial(\lambda_i u_j)}{\partial x_j} + \lambda_j \frac{\partial u_j}{\partial x_i} - \nu\nabla^2\lambda_i + \lambda_{i+3} = 0 \quad (4A)$$

$$\delta P: \frac{\delta F}{\delta P} = \frac{\partial \lambda_i}{\partial x_i} \quad (4B)$$

$$\delta\Pi_1: \frac{\delta F}{\delta \Pi_1} = \frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3} \quad (4C)$$

$$\delta\Pi_2: \frac{\delta F}{\delta \Pi_2} = \frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3} \quad (4D)$$

Here we have obtained a composite VP with the composite functional I in Eq.(3) for the composite system covering both the primary system Eqs.(1) & (2) and the adjoint system Eqs.(4A-4D) involving the adjoint (or called the dual) variables λ_k and the primary variables u_i, P, Π_1 & Π_2. To identify F and reduce the number of unknown variables, one possible simple way is to put:

$$\frac{\delta F}{\delta P} = P, \quad \frac{\delta F}{\delta \Pi_1} = \Pi_1, \quad \frac{\delta F}{\delta \Pi_2} = \Pi_2, \quad \frac{\delta F}{\delta u_i} = 0, \quad (5)$$

so that $F = (P^2 + \Pi_1{}^2 + \Pi_2{}^2)/2$ (6)

which apparently satisfies the integrability conditions for the function F. [16]

Upon substituting Eq.(6), the adjoint Eqs.(4A) ~ (4D) can be simplified to the following form

$$\lambda_{i+3} = \frac{\partial \lambda_i}{\partial t} + \frac{\partial(\lambda_i u_j)}{\partial x_j} - \lambda_j \frac{\partial u_j}{\partial x_i} + \nu\nabla^2\lambda_i \quad (7A)$$

$$P = \frac{\partial \lambda_i}{\partial x_i} \quad (7B)$$

$$\Pi_1 = \frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3} \quad (7C)$$

$$\Pi_2 = \frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3} \quad (7D)$$

Substituting Eqs.(7C)-(7D)in Eq.(1) results in

$$u_1 = \frac{\partial}{\partial x_3}\left(\frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3}\right) \quad (7E)$$

$$u_2 = \frac{\partial}{\partial x_3}\left(\frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3}\right) \quad (7F)$$

$$-u_3 = \frac{\partial}{\partial x_1}\left(\frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3}\right) \quad (7G)$$

It is interesting to point out that Eqs.(7A)~(7D) are something like the Clebsch transformation[1,4] and will be employed later on for removing λ_{i+3}, P, Π_1 and Π_2.

Substituting the Euler equations(1) in Eq.(3) to eliminate u_i, and at the same time $\lambda_4 \sim \lambda_6$ are automatically removed, so we obtain the functional I_I of the following variational principle:

Mixed VP-I: $\delta I_I = 0$

$$I_I(P, \Pi_1, \Pi_2, \lambda_i) = \int_t \iiint_{(\Omega)} \{\frac{1}{2}(P^2 + \Pi_1{}^2 + \Pi_2{}^2) + \lambda_i(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} - \nu\nabla^2 u_i - f_i)\} \mathrm{dVdt} \quad (8)$$

with u_i represented by Eq.(1).

It is easy to show that if we take the first variation of the functional I_I and set it equal to zero, we obtain the following Euler equations:

$\delta\lambda_i$: Eq.(2)

$$\delta P: \ P = \frac{\partial \lambda_i}{\partial x_i} \quad (8A)$$

$$\delta\Pi_1 : \quad \Pi_1 = \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \qquad (8B')$$

$$\delta\Pi_2 : \quad \Pi_2 = \frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_2} \qquad (8C')$$

where H_i stand for the right hand side of Eq.(7A), so that Eq.(7A) takes the form: $H_i = -\lambda_{3+i}$, with which Eqs.(8B')&(8C') can be transformed into Eqs.(7C) &(7D)and in turn into Eq.(1) via Eqs.(7E)-(7G). Thus eventually we have the Euler equations of VP-I: Eqs.(1)&(2).

Using this VP-I, we can obtain directly the numerical solution of λ_i, P, Π_1 and Π_2 by the finite element method (FEM). For practical use, however, it is also required to have the u_i fields calculated. This can be simply done by means of Eq.(1).

If we insert Eq.(7B) in Eq.(8) to eliminate P, we obtain the following new simpler variational principles: **Mixed VP-II** : $\delta I_{II} = 0$ and

$$I_{II}(\Pi_1, \Pi_2, \lambda_i) =$$

$$= \int_t \iiint_{(\Omega)} \left\{ \left[\left(\frac{\partial \lambda_i}{\partial x_i}\right)\left(\frac{\partial \lambda_j}{\partial x_j}\right) + \Pi_1{}^2 + \Pi_2{}^2 \right] \Big/ 2 + \lambda_i \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial^2 \lambda_j}{\partial x_i \partial x_j} - \nu\nabla^2 u_i - f_i \right) \right\} dVdt \qquad (9)$$

with u_i represented by Eq.(1).

Taking the first variation of the functional I_{II} and set it equal to zero, we obtain the following Euler equations:

$$\delta\lambda_i : \quad \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial^2 \lambda_j}{\partial x_i \partial x_j} - \nu\nabla^2 u_i - f = 0 \qquad (9A)$$

which, upon inserting Eq.(7B), converts into Eq.(2).

$$\delta\Pi_1 : \quad \Pi_1 = \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \qquad (8B')$$

$$\delta\Pi_2 : \quad \Pi_2 = \frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_2} \qquad (8C')$$

which convert into Eq.(1) as shown above.

After the Π_1, Π_2, λ_i fields have been obtained from the VP-II by FEM, the practically most interesting velocities u_i and pressure P can be calculated by Eqs.(1) & (7B).

Since I_I and I_{II} contain both primary and dual variables, they are called mixed variational principles:

Mixed VP-III($\delta I_{III} = 0$): Still another mixed variational principle can be derived by using Eq.(7A) to eliminate the adjoint variables λ_4, λ_5 & λ_6 in Eqs.(7C) & (7D), giving

$$\left.\begin{aligned}
\Pi_1 &= \left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} + \nu\nabla^2\right)\left(\frac{\partial \lambda_3}{\partial x_1} - \frac{\partial \lambda_1}{\partial x_3}\right) \\
&+ \left(\frac{\partial \lambda_3}{\partial x_j} + \frac{\partial \lambda_j}{\partial x_3}\right)\frac{\partial u_j}{\partial x_1} - \left(\frac{\partial \lambda_1}{\partial x_j} + \frac{\partial \lambda_j}{\partial x_1}\right)\frac{\partial u_j}{\partial x_3} \\
\Pi_2 &= \left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} + \nu\nabla^2\right)\left(\frac{\partial \lambda_3}{\partial x_2} - \frac{\partial \lambda_2}{\partial x_3}\right) \\
&+ \left(\frac{\partial \lambda_3}{\partial x_j} + \frac{\partial \lambda_j}{\partial x_3}\right)\frac{\partial u_j}{\partial x_2} - \left(\frac{\partial \lambda_2}{\partial x_j} + \frac{\partial \lambda_j}{\partial x_2}\right)\frac{\partial u_j}{\partial x_3}
\end{aligned}\right\} \qquad (10)$$

The corresponding functional $I_{III}(u_i, \lambda_i)$ has the followingform

$$\begin{aligned}
I_{III}(u_i, \lambda_i) = \int_t \iiint_{(\Omega)} \Bigg\{ & \frac{1}{2}\left[\left(\frac{\partial \lambda_i}{\partial x_i}\right)\left(\frac{\partial \lambda_j}{\partial x_j}\right) + \Pi_1{}^2 + \Pi_2{}^2 \right] \\
& + \lambda_i \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial^2 \lambda_j}{\partial x_i \partial x_j} - \nu\nabla^2 u_i - f_i \right) \\
& + \left(\frac{\partial \lambda_1}{\partial t} + \frac{\partial(\lambda_1 u_j)}{\partial x_j} - \lambda_j \frac{\partial u_j}{\partial x_1} + \nu\nabla^2 \lambda_1 \right)\left(u_1 - \frac{\partial \Pi_1}{\partial x_3} \right) \\
& + \left(\frac{\partial \lambda_2}{\partial t} + \frac{\partial(\lambda_2 u_j)}{\partial x_j} - \lambda_j \frac{\partial u_j}{\partial x_2} + \nu\nabla^2 \lambda_2 \right)\left(u_2 - \frac{\partial \Pi_2}{\partial x_3} \right) \\
& + \left(\frac{\partial \lambda_3}{\partial t} + \frac{\partial(\lambda_3 u_j)}{\partial x_j} - \lambda_j \frac{\partial u_j}{\partial x_3} + \nu\nabla^2 \lambda_3 \right) \\
& \left(u_3 + \frac{\partial \Pi_1}{\partial x_1} + \frac{\partial \Pi_2}{\partial x_2} \right) \Bigg\} \mathrm{dVdt}
\end{aligned} \qquad (11)$$

where Π_1 & Π_2 are represented by Eqs.(10).

From $\delta I_{III} = 0$ the Euler equations of the VP-III result:

$\delta\lambda_i$: Eq.(2)

δu_i: Eq.(1)

If the pressure field is of interest, it can be readily calculated from Eq.(7B) by simple differentiation.

2.3 A Dual VP

It is still possible to eliminate Π_1 & Π_2 in Eq. (9) via inserting Eqs.(7C) & (7D), leading to the following

Dual VP-IV: $\delta I_{IV} = 0$ and

$$I_{IV}(\lambda_k) = \int_t \iiint_{(\Omega)} \left\{ \left[\left(\frac{\partial \lambda_i}{\partial x_i} \right) \left(\frac{\partial \lambda_j}{\partial x_j} \right) + \left(\frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3} \right)^2 + \left(\frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3} \right)^2 \right] \Big/ 2 + \lambda_i \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial^2 \lambda_j}{\partial x_i \partial x_j} - \nu \nabla^2 u_i - f_i \right) \right\} dVdt \tag{12}$$

with u_i represented by

$$\left.\begin{aligned} u_1 &= \frac{\partial \Pi_1}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3} \right) \\ u_2 &= \frac{\partial \Pi_2}{\partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3} \right) \\ -u_3 &= \frac{\partial \Pi_1}{\partial x_1} + \frac{\partial \Pi_2}{\partial x_2} = \nabla^2 \lambda_6 - \frac{\partial}{\partial x_3}\left(\frac{\partial \lambda_4}{\partial x_1} + \frac{\partial \lambda_5}{\partial x_2} + \frac{\partial \lambda_6}{\partial x_3} \right) \end{aligned}\right\} \tag{13}$$

It can be easily shown that from $\delta I_{IV} = 0$ we obtain the following Euler's equations:

$\delta\lambda_i$: Eq.(9A), which converts into Eq.(2) via Eq.(7B).

$$\delta\lambda_4 : \quad \frac{\partial}{\partial x_3}\left(\frac{\partial(\lambda_6 + H_3)}{\partial x_1} - \frac{\partial(\lambda_4 + H_1)}{\partial x_3} \right) = 0 \tag{13A}$$

$$\delta\lambda_5 : \quad \frac{\partial}{\partial x_3}\left(\frac{\partial(\lambda_6 + H_3)}{\partial x_2} - \frac{\partial(\lambda_5 + H_2)}{\partial x_3} \right) = 0 \tag{13B}$$

$$\delta\lambda_6 : \quad \frac{\partial}{\partial x_1}\left(\frac{\partial(\lambda_4 + H_1)}{\partial x_3} \right) + \frac{\partial}{\partial x_3}\left(\frac{\partial(\lambda_5 + H_2)}{\partial x_2} \right) - \tilde{\nabla}^2(\lambda_6 + H_3) = 0 \tag{13C}$$

where $\tilde{\nabla}^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

Using Eqs.(13), (8B') &(8C'), Eqs.(13A)-(13C) can be transformed into Eq.(1).

Based on VP-IV, we can obtain directly the λ_k-fields by FEM. The practically important u_i and P can then be computed simply by means of Eq.(13) and Eq.(7B) respectively.

3 Derivation of The Second VP Family

A second VP family can be derived by identifying the undetermined function F in an alternative way. For this purpose let us return to Eqs.(4A) ~ (4D) and set

$$\frac{\delta F}{\delta P} = u_3, \quad \frac{\delta F}{\delta \Pi_1} = u_1, \quad \frac{\delta F}{\delta \Pi_2} = u_2, \tag{14A}$$

resulting in

$$F = u_1\Pi_1 + u_2\Pi_2 + u_3\Pi_3 \tag{14B}$$

and

$$\frac{\delta F}{\delta u_1} = \Pi_1, \quad \frac{\delta F}{\delta u_2} = \Pi_2, \quad \frac{\delta F}{\delta u_3} = \Pi_3 \tag{14C}$$

where for convenience a new symbol Π_3 is introduced instead of P. Obviously, Eqs.(14A, B, C) satisfy the integrability conditions of Ref. [16].

Using Eqs.(14A, B, C) the Euler equations (4A) ~ (4D) take the following form:

$$u_1 = \frac{\partial \lambda_6}{\partial x_1} - \frac{\partial \lambda_4}{\partial x_3}, \quad u_2 = \frac{\partial \lambda_6}{\partial x_2} - \frac{\partial \lambda_5}{\partial x_3}, \quad u_3 = \frac{\partial \lambda_i}{\partial x_i} \tag{15A}$$

$$\Pi_i = \frac{\partial \lambda_i}{\partial t} + u_j \frac{\partial \lambda_i}{\partial x_j} - \lambda_j \frac{\partial u_j}{\partial x_i} + \nu \nabla^2 \lambda_i - \lambda_{i+3} \tag{15B}$$

Thus we obtain the following VP:

Mixed VP-V:

Using the Euler Eqs.(1) to eliminate u_i in Eq.(3), we obtain a new VP: $\delta I_V = 0$ and

$$I_V(\Pi_i,\lambda_i)=\int_t\iiint_{(\Omega)}\left\{\frac{1}{2}\frac{\partial}{\partial x_3}\left(\Pi_1^2+\Pi_2^2\right)\right.$$
$$-\left(\frac{\partial\Pi_1}{\partial x_1}+\frac{\partial\Pi_2}{\partial x_2}\right)\Pi_3+\lambda_i\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right.$$
$$\left.\left.+\frac{\partial\Pi_3}{\partial x_i}-\nu\nabla^2u_i-f_i\right)\right\}dVdt \qquad (16)$$

with u_i represented by Eq.(1).

It is interesting enough to note that the functional I_V in Eq.(16) can be simplified a lot by dropping the first term under integral: $\frac{1}{2}\frac{\partial}{\partial x_3}\left(\Pi_1^2+\Pi_2^2\right)$, due to the fact that its variation vanishes identically.

From $\delta I_V=0$ we obtain the Euler equations of VP-V as follows:

$\delta\lambda_i$: Eq.(2)

$$\delta\Pi_1:\ \frac{\partial\Pi_3}{\partial x_1}+\frac{\partial H_3}{\partial x_1}-\frac{\partial H_1}{\partial x_3}=0 \qquad (16A)$$

$$\delta\Pi_2:\ \frac{\partial\Pi_3}{\partial x_2}+\frac{\partial H_3}{\partial x_2}-\frac{\partial H_2}{\partial x_3}=0 \qquad (16B)$$

$$\delta\Pi_3:\ \frac{\partial\Pi_1}{\partial x_1}+\frac{\partial\Pi_2}{\partial x_2}+\frac{\partial\lambda_i}{\partial x_i}=0 \qquad (16C)$$

Using Eq.(15B)(i.e. $\Pi_i=-H_i-\lambda_{i+3}$) and Eq.(15A), Eqs.(16A) ~ (16C) can be rewritten as Eq.(1). Thus, eventually the Euler equations of the VP-V are Eqs.(2) and (1).

Applying the FEM to VP-V, we obtain directly the λ_i, Π_1, Π_2 and $\Pi_3(=P)$ fields over the solution domain under study. After that the corresponding velocity distribution u_i can be computed according to Eq.(1).

Furthermore, if Eq. (15B) is substituted in Eq. (16) to eliminate Π_i and Eq.(1) is used in turn to eliminate Π_i, we obtain another **Dual VP-VI:** $\delta I_{VI}(\lambda_k)=0$ with u_i represented by Eq.(1).

4 Concluding Remarks

Two families of VPs for the full N-S equations of 3-D unsteady flow have been systematically established herein for the first time. Thus, a new rigorous theoretical basis for the finite element analysis of viscous flow, especially for the direct numerical simulation (DNS)of turbulent flow has been founded.

It should be pointed out that as a rule, in almost all existing VP in mechanics the physical meaning of the functional is some kind of energy[1,16], the functionals of all VP derived herein, in contrast, do not have energy meaning since neither P^2 nor $u_i\Pi_i$ have the dimension of energy. Perhaps, this might be just the origin of the difficulty associated with the derivation of the VP for the full N-S equations. Fortunately, just recently we have succeeded in generating an alternative dual VP family via Friedrichs' involutory transformation[20]whose functionals do have physical meaning of kinetic energy. These dual VP can be easily generalized to unsteady flow as well.

Evidently, our success in deriving VP for the N-S equations is strongly due to the following two key measures: 1)introducing the undetermined function F in the trial functionals; 2)introducing the pseudo-stream functions. Finally, the present approach and results can be extended to viscous flows in a rotating system (e.g. in turbomachines) as well as to compressible viscous flows.

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