Network-delay-dependent $\mathcal{H}_\infty$ control of systems with input saturation

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Abstract: This paper addresses the problem of designing a network-delay-dependent switching controller that achieves the $\mathcal{H}_\infty$ disturbance-rejection performance under an $\mathcal{L}_\infty$ performance representing componentwise input saturation. To design such controller, this paper first builds up the conditions for set invariance, and then incorporates these conditions in the synthesis of dynamic state-feedback $\mathcal{H}_\infty$ control. The proposed design conditions are characterized in terms of linear matrix inequalities with one prescribed scalar.

Key–Words: Network delay, Input saturation, $\mathcal{H}_\infty$ performance

1 Introduction

The technologies on multi-accessible communication networks, such as Ethernet with flexibility, cost effectiveness, speed improvement, and distributiveness, have been recently received considerable attention in the area of networked control systems (NCSs) since multi-accessible communication networks are very useful for data transmission linkage in control applications. However, since the multi-accessibility causes the communication networks to suffer from random networked-induced delays that deteriorate the stability and control performance of closed-loop control systems, one needs to definitely handle the delays when implementing a feedback control loop closed through multi-accessible communication networks. Thus, numerous investigations and research efforts have been undertaken to deal with the delays (see *e.g.*, [7]–[11], and references therein).

In more practical application for such systems, one necessarily needs to address the input saturation during the control design procedure since every physical actuator is subject to the saturation that deteriorates the stability of the control applications. However, to the best of our knowledge, there has been yet no results of taking into account the saturation of actuators in the process of constructing the NCSs. Of course, for the traditional point-to-point communication network, various research results of handling explicitly the input saturation have been already published in the system and control literature, particularly of which several important results of handling directly the input saturation have recently appeared well in [1], [4]–[6], and references therein. In this paper, we shall use the polytopic representation method, proposed first in [4], to handle the input saturation nonlinearity, which allows high gain control to be used for stabilization.

In order to address explicitly the effects of both data-transmission delay and loss of data, first, we employ a reliable transport protocol that guarantees data delivery and supports that transmitted data have their time-stamp information. Based on such a protocol, we propose a discrete-time system over asymmetric path-delay configurations (SOAP) on the high-speed networks, which has the same structure as that of [7] except for the point that a saturator is inserted between the controller and the communication networks, depicted in Fig. 1-(a). As mentioned in [7], the SOAP has two different paths sharing a reliable transport protocol; one path delivers constant-delayed data to the destination by using the FIFO (First-In-First-Out) data buffer, and the other path delivers data with their time-stamp information to the destination. In this framework, we shall develop a systematic methodology for designing an $\mathcal{H}_\infty$ control for NCSs subject to input saturation via a deterministic approach. To this end, we first propose a dynamic state-feedback quasi-linear parameter-varying (QLPV) control law dependent on the previous mode which denotes one of statuses that a switching controller can belong to. Based the control law, we formulate the conditions for set invariance in terms of LMIs with one prescribed scalar. And then, we use the obtained conditions in constructing a network-delay-dependent $\mathcal{H}_\infty$ controller which achieves the maximal disturbance rejection.

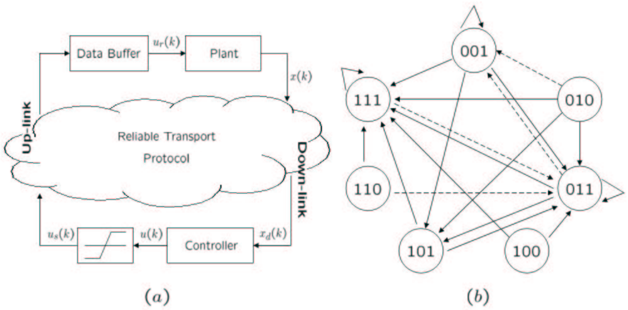

Figure 1: (a) The SOAP with a saturating controller. (b) A mode transition diagram ($d_{\max} = 2$).

2 Preliminaries

Consider the following linear time-invariant (LTI) system of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u_r(k), \\ z(k) &= Cx(k) + D_1 w(k) + D_2 u_r(k), \end{aligned} \quad (1)$$

where $x(k) \in \mathcal{R}^n$, $u_r(k) \in \mathcal{R}^m$, $w(k) \in \mathcal{R}^p$ and $z(k) \in \mathcal{R}^q$ denote the state, the input, the disturbance and the performance output, respectively. Here, it is assumed that the disturbance $w(k)$ is unknown but belongs to $\mathcal{W}_\delta := \left\{ w \in \mathcal{R}^p \mid w^T(k)w(k) \le \delta, \ \delta \ge 0, \ \forall k \ge 0 \right\}$.

Before going ahead, we make the same three assumptions (A1), (A2), and (A3) as did [7]. As mentioned in [7], in the down-link (see Fig. 1-(a)), since the controller does not exactly know when the input $u_s(k)$ acts on the plant, we shall force the down-link delay to be fixed into its bound value d_M, i.e., $u_r(k) = u_s(k - d_M)$. Contrary to the down-link, in the up-link (see Fig. 1-(a)), we shall use the real-time information on the up-link delay sequences delivered to the controller (time-stamp information). Refer [7] for the detailed explanation on the proposed SOAP structure. With the above settings, the resulting system model in the controller point of view is given as follows:

$$\tilde{x}(k+1) = \tilde{A}\, \tilde{x}(k) + \tilde{B}_1\, w(k) + \tilde{B}_2\, u_s(k), \quad (2)$$

$$z(k) = \tilde{C}\, \tilde{x}(k) + \tilde{D}\, w(k), \quad (3)$$

$$x_d(k) = \tilde{E}(k)\, \tilde{x}(k), \quad (4)$$

$$u_s(k) = \text{sat}(u(k), \bar{u}), \quad (5)$$

where $\tilde{x}(k) := [x^T(k) \mid x^T(k-1) \cdots x^T(k - d_M) \mid u_s^T(k-1) \cdots u_s^T(k - d_M)]^T \in \mathcal{R}^t$, $t = n + (n+m)d_M$, $x_d(k) \in \mathcal{R}^n$, $u(k) \in \mathcal{R}^m$ and $\bar{u} \in \mathcal{R}^m$ denote the augmented state, the delayed state, the raw control input and the saturation level, respectively, and the matrices are defined as

$$[\tilde{A}|\tilde{B}_1|\tilde{B}_2] = \left[\begin{array}{ccccc|c|c} A & 0 & 0 & 0 & B_2 & B_1 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 & 0 \end{array} \right], \quad (6)$$

$$\tilde{C} = \left[\begin{array}{ccccc} C & 0 & 0 & 0 & D_2 \end{array} \right], \ \tilde{D} = D_1, \quad (7)$$

$$\tilde{E}(k) = \left[\begin{array}{c|c} \text{diag}(\Phi_{k0}, \cdots, \Phi_{km}) & 0 \end{array} \right], \quad (8)$$

where if $x(k-r)$ is avilable at time k, Φ_{kr} is set to identity matrix, and otherwise, Φ_{kr} is set to zero matrix. $x_d(k)$ is determined by (4) with the help of the time-stamp information. By the basic characteristics of the SOAP, we can determine $(2^{d_M+1} - 1)$ different $\tilde{E}(k)$ for a given d_M. We shall henceforth call each status corresponding to $\tilde{E}(k)$ a mode, *say*, $m(k)$. The $m(k)$ will be expressed as $m(k) = (b_0 b_1 \cdots b_{d_M})_2$, where $(\cdot)_2$ means the binary representation of $m(k)$. The r-th bit, *say*, b_r, is set to 1 if the r-delayed state, $x(k-r)$, is available, otherwise, the bit is set to 0. The mode $m(k)$, hence, belongs to a set $\mathcal{M} := \{m \in \mathcal{R} \mid m = 1, 2, \cdots, 2^{d_{\max}+1} - 1\}$. Besides, based on the second assumption (A2), we can uniquely determine a set of transitions, *say*, $\mathcal{S}$, only if d_M is determined: $\mathcal{S} := \{(m(k), m(k-1)) \mid$ all possible transition pairs yielding (A1) and (A2) for $m(k) \in \mathcal{M}, m(k-1) \in \mathcal{M}, \forall k\}$, For example, if d_M is 1, we have three modes, $(10)_2$, $(01)_2$, and $(11)_2$, and a set of transitions $\mathcal{S} = \{(1,1), (1,2), (1,3), (3,1), (3,2), (3,3)\}$. In the case of $d_M = 2$, possible modes and transitions are shown in Fig. 1(b). One mode may transit to other mode with no received data, which is represented as the dotted lines; other kinds of transitions are represented as the solid lines. In this paper, we shall directly handle the input saturation nonlinearity by using the following lemma proposed in [4].

Lemma 1 *Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose that $|v_r| \le \bar{u}_r$ for all $r = 1, \cdots, m$, where v_r and $\bar{u}_r$ denote the r-th element of $v \in \mathcal{R}^m$ and $\bar{u} \in \mathcal{R}^m$, respectively. Then*

$$\text{sat}(u, \bar{u}) = \sum_{\ell=1}^{2^m} \alpha_\ell \left(D_\ell u + D_\ell^- v \right), \ \sum_{\ell=1}^{2^m} \alpha_\ell = 1, \quad (9)$$

where $\alpha_\ell \ge 0$ and D_ℓ denote all elements of $\mathcal{D}$, and $D_\ell^- = I - D_\ell$.

Consider a previous mode (PM)-dependent dynamic quasi-linear parameter varying (QLPV) control law which switches itself depending on its previous and current modes:

$$x_c(k+1) = F_{ji}(k)x_c(k) + G_{ji}(k)\tilde{E}_j\tilde{x}(k), \quad (10)$$
$$u(k) = H_{ji}x_c(k) + J_{ji}\tilde{E}_j\tilde{x}(k) \quad (11)$$

subject to

$$[F_{ji}(k)\ G_{ji}(k)] = \sum_{\ell=1}^{2^m} \alpha_\ell(k)\left[F_{ji}^\ell\ G_{ji}^\ell\right], \quad (12)$$

where $\alpha_\ell(k)$ denote the interpolation coefficients at time k in (9), the subscripts j and i stand for $m(k)$ and $m(k-1)$, respectively, and $x_c(k) \in \mathcal{R}^t$ denotes the controller state. In order to use Lemma 1 in representing the input saturation nonlinearity (5), we shall employ an auxiliary PM-dependent control input $v(k)$ as follows:

$$v(k) = K_{ji}x_c(k) + L_{ji}\tilde{E}_j\tilde{x}(k) = V_{ji}\bar{x}(k), \quad (13)$$

where $V_{ji} := \left[L_{ji}\tilde{E}_j\ K_{ji}\right]$. Thus, if $\bar{x}(k)$, for all $k \geq 0$, belongs to $\mathcal{L}(V_{ji})$ defined as

$$\mathcal{L}(V_{ji}) := \left\{\bar{x} \in \mathcal{R}^t \mid -\bar{u} \leq V_{ji}\bar{x}(k) \leq \bar{u}\right\}, \quad (14)$$

then

$$\mathrm{sat}\left(U_{ji}\bar{x}(k), \bar{u}\right) = \sum_{\ell=1}^{2^m} \alpha_\ell(k)\left\{D_\ell U_{ji} + D_\ell^- V_{ji}\right\}\bar{x}(k),$$

where $U_{ji} := \left[J_{ji}\tilde{E}_j\ H_{ji}\right]$. Accordingly, the resulting closed-loop system subject to $\bar{x}(k) \in \mathcal{L}(V_{ji})$, for all $k \geq 0$, can be rewritten as

$$\bar{x}(k+1) = \bar{A}_{ji}(k)\bar{x}(k) + \bar{B}w(k), \quad (15)$$
$$z(k) = \bar{C}\bar{x}(k) + \bar{D}w(k), \quad (16)$$

where the matrices are defined as

$$\bar{A}_{ji}(k) = \sum_{\ell=1}^{2^m} \alpha_\ell(k)\bar{A}_{ji}^\ell,\ \sum_{\ell=1}^{2^m} \alpha_\ell(k) = 1,\ \alpha_\ell(k) \geq 0,$$
$$\bar{A}_{ji}^\ell := \begin{bmatrix} \tilde{A} + \tilde{B}_2\left(D_\ell J_{ji} + D_\ell^- L_{ji}\right)\tilde{E}_j & \tilde{B}_2\left(D_\ell H_{ji} + D_\ell^- K_{ji}\right) \\ G_{ji}^\ell \tilde{E}_j & F_{ji}^\ell \end{bmatrix},$$
$$\bar{B}^T := \begin{bmatrix} \tilde{B}_1^T & 0 \end{bmatrix}^T,\ \bar{C} = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix},\ \bar{D} = \tilde{D}.$$

3 Main Results

First of all, we shall find the conditions for obtaining the ellipsoidal sets $\mathcal{E}(P_i)$ such that, for all $k \geq 0$, $i \in \mathcal{M}$ and $w \in \mathcal{W}_\delta$,

$$\psi(k, \bar{x}(0), w) \in \mathcal{E}(P_i),\ \forall \bar{x}(0) \in \mathcal{E}(P_i), \quad (17)$$

where $\psi(k, \bar{x}(0), w)$ denotes the state trajectory of the closed-loop system and $\mathcal{E}(P_i)$ denote PM-dependent ellipsoidal sets defined as, for all $i \in \mathcal{M}$,

$$\mathcal{E}(P_i) := \left\{\bar{x} \in \mathcal{R}^t \mid \bar{x}^T P_i \bar{x} \leq 1,\ P_i > 0\right\}. \quad (18)$$

In the following lemma, we present the conditions for obtaining the ellipsoidal sets $\mathcal{E}(P_i)$ with the property (17).

Lemma 2 *Let $\delta \geq 0$ be given. Suppose that there exist $0 \leq \lambda_1 \leq 1$ and $\bar{P}_i > 0$, $i \in \mathcal{M}$, such that, for all $(j,i) \in \mathcal{S}$ and $\ell \in [1, 2^m]$,*

$$0 \leq \begin{bmatrix} \lambda_1 P_i & 0 & (*) \\ 0 & (1/\delta)(1-\lambda_1)I & (*) \\ \bar{A}_{ji}^\ell & \bar{B} & \bar{P}_j \end{bmatrix}, \quad (19)$$
$$\mathcal{E}(P_i) \subset \mathcal{L}(V_{ji}), \quad (20)$$

where $\bar{P}_i := P_i^{-1}$. Then there exist the ellipsoidal sets $\mathcal{E}(P_i)$ such that, for $i \in \mathcal{M}$ and $w \in \mathcal{W}_\delta$,

$$\psi(k, \bar{x}(0), w) \in \mathcal{E}(P_i),\ \forall \bar{x}(0) \in \mathcal{E}(P_i). \quad (21)$$

■

We denote the $\mathcal{H}_\infty$ norm boundedness of the transfer function from w to z, T_{zw}, as $||T_{zw}||_\infty < \gamma$, i.e., for all nonzero $w(k) \in \mathcal{L}_{2+}$,

$$||z(k)||_2 < \gamma^2 ||w(k)||_2, \quad (22)$$

where the upper bound γ is in inverse proportion to the disturbance rejection capability. In this paper, we shall solve the following problem of minimizing the upper bound γ so as to construct a PM-dependent dynamic QLPV controller which achieves the maximal disturbance rejection capability:

$$\min \gamma \text{ subject to } (19), (20), \text{ and } (22), \quad (23)$$

where the conditions (19) and (20) make the state trajectories remain inside $\mathcal{E}(P_i) \subset \mathcal{L}(V_{ji})$, and thus the transition of the state $\bar{x}(k)$ is always determined by the closed-loop system (15).

In the following proposition, we propose the method of designing a PM-dependent $\mathcal{H}_\infty$ controller via LMI approach.

Proposition 3 *Let $\delta \geq 0$ be given. For a prescribed value $0 \leq \lambda_1 \leq 1$, suppose that there exist X_i, $\bar{X}_i$, $\Psi^{\ell}_{1,ji}$, $\Psi_{2,ji}$, $\Psi_{3,ji}$, $\Pi^{\ell}_{ji}(1,2)$, J_{ji}, L_{ji}, and Γ that are solutions of the following optimization problem:*

$$\gamma^* = \min \gamma \tag{24}$$

subject to

$$0 \leq \begin{bmatrix} \lambda_1 X_i & (*) & (*) & (*) & (*) \\ \lambda_1 I & \lambda_1 \bar{X}_i & (*) & (*) & (*) \\ 0 & 0 & \rho I & (*) & (*) \\ \Pi^{\ell}_{ji}(1,1) & \Pi^{\ell}_{ji}(1,2) & X_j \tilde{B}_1 & X_j & (*) \\ \Pi^{\ell}_{ji}(2,1) & \Pi^{\ell}_{ji}(2,2) & \tilde{B}_1 & I & \bar{X}_j \end{bmatrix},$$

$$0 \leq \begin{bmatrix} \Gamma & L_{ji}\tilde{E}_j & \Psi_{3,ji} \\ (*) & X_i & I \\ (*) & (*) & \bar{X}_i \end{bmatrix}, \ \Gamma_{rr} \leq \bar{u}_r^2,$$

$$0 \leq \begin{bmatrix} X_i & (*) & (*) & (*) & (*) & (*) \\ I & \bar{X}_i & (*) & (*) & (*) & (*) \\ 0 & 0 & \gamma^2 I & (*) & (*) & (*) \\ \Pi^{\ell}_{ji}(1,1) & \Pi^{\ell}_{ji}(1,2) & X_j \tilde{B}_1 & X_j & (*) & (*) \\ \Pi^{\ell}_{ji}(2,1) & \Pi^{\ell}_{ji}(2,2) & \tilde{B}_1 & I & \bar{X}_j & (*) \\ \tilde{C} & \tilde{C}\bar{X}_i & \tilde{D} & 0 & 0 & I \end{bmatrix},$$

where $\rho := (1/\delta)(1-\lambda_1)$, and Γ_{rr} denotes the r-th diagonal element of Γ,

$$\begin{aligned} \Pi^{\ell}_{ji}(1,1) &:= X_j \tilde{A} + \Psi^{\ell}_{1,ji}\tilde{E}_j, \\ \Pi^{\ell}_{ji}(1,2) &:= X_j \tilde{A}\bar{X}_i + X_j \tilde{B}_2 \left(D_\ell H_{ji} + D_\ell^- K_{ji}\right)\bar{Y}_i^T \\ &\quad + + \Psi^{\ell}_{1,ji}\tilde{E}_j\bar{X}_i + Y_j F^{\ell}_{ji}\bar{Y}_i^T, \\ \Pi^{\ell}_{ji}(2,1) &:= \tilde{A} + \tilde{B}_2\left(D_\ell J_{ji} + D_\ell^- L_{ji}\right)\tilde{E}_j, \\ \Pi^{\ell}_{ji}(2,2) &:= \tilde{A}\bar{X}_i + \tilde{B}_2\left(D_\ell \Psi_{2,ji} + D_\ell^- \Psi_{3,ji}\right), \\ \Psi^{\ell}_{1,ji} &:= X_j \tilde{B}_2\left(D_\ell J_{ji} + D_\ell^- L_{ji}\right) + Y_j G^{\ell}_{ji}, \\ \Psi_{2,ji} &:= J_{ji}\tilde{E}_j\bar{X}_i + H_{ji}\bar{Y}_i^T, \\ \Psi_{3,ji} &:= L_{ji}\tilde{E}_j\bar{X}_i + K_{ji}\bar{Y}_i^T. \end{aligned}$$

Then closed-loop system is asymptotically stable in the absence of disturbances, and $||z(k)||_2 \leq \gamma^*||w(k)||_2$ holds in the presence of disturbances. ∎

4 Numerical Example

To verify the performance of the proposed control algorithm, we consider a discrete linear time-invariant (LTI) plant model (1) with the following system matrices:

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right] = \left[\begin{array}{cc|c|c} 1.0 & -0.5 & 0.1 & 1.0 \\ 1.0 & 0.5 & 0.5 & 0.5 \\ \hline 0.3 & 0.6 & 1.0 & 0.2 \end{array}\right].$$

Table 1: Disturbance rejection capability

d_M	$\bar{u}$	γ^*	$\bar{u}$	γ^*
0	1	1.3641	3	1.2505
1	1	1.4723	3	1.4343
2	1	2.6027	3	1.8046

For $\delta = 0.6$ and $\bar{u} = 1$ (or $\bar{u} = 3$), we solve the optimization problem in Proposition 3 to obtain the upper bound γ^* where the prescribed value λ_1 is tuned between 0 and 1. Simulations are performed at various d_M values from $d_M = 0$ to $d_M = 2$, where random delay sequences are generated with the same method as did in [7].

5 Concluding Remarks

In this paper, we addressed the problem of designing an $\mathcal{H}_\infty$ control for networked control systems (NCSs) with the effects of both the network-induced delay and the input saturation. To design such control, we first found the conditions for set invariance, characterized by LMIs with one prescribed scalar, and then used these conditions for designing a dynamic state-feedback $\mathcal{H}_\infty$ control dependent on previous mode. We verified the performance of the proposed control algorithm via a numerical example.

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