Fusion and Consistency of Incomplete Fuzzy Preferences

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Abstract: - The method proposed by Herrera et al. [11] devises a consistent preference relation that is restricted by the set of $n-1$ values $\{p_{12}, p_{23},..., p_{n-1n}\}$. Therefore, for convenience and flexibility, the following uses the incomplete fuzzy preference relation with the least judgments (that is, $n-1$ judgments) to develop a simple and practical method for constructing a consistent complete fuzzy preference relation in which experts can compare any row, column or diagonal.

Key-Words: - Incomplete fuzzy preference relations; consistency; decision making; fusion

1 Introduction

Preference relations is widely employed in most decision processes [1,3,10,15]. A well-known approach that can effectively deal with decision problems is the Analytic Hierarchy Process (AHP) proposed by Saaty [8]. The AHP methodology involves separating a complex decision issue into elemental problems to establish a hierarchical model. When the decision problem is divided into smaller constituent parts in a hierarchy, pairwise comparisons of the relative importance of elements are performed in each level of the hierarchy to establish a set of weights or priorities. Although AHP is widely employed in diverse fields [2,5,6,7,10,16], inconsistency occurs given increasing hierarchies of criteria or alternatives. Additionally, each of these preference relations necessitates the completion of all $\frac{n(n-1)}{2}$ judgements throughout its top triangular portion. However, it is sometimes difficult to yield such a complete preference relation, particularly for high order preference relations. To resolve this dilemma, Herrera-Viedma et al. [11] presented the consistent fuzzy preference relations for facilitating decision-making, thus enhancing its effectiveness and accuracy of selections. However, in this method the construction of a consistent preference relation is restricted by the set of $n-1$ values $\{p_{12}, p_{23},..., p_{n-1n}\}$.

For convenience and flexibility, this study developed a method, based on the consistent incomplete fuzzy preference relation involving lease judgements, for constructing a consistent complete fuzzy preference relation using the multiplicative transitivity property. The increasing complexity and uncertainty of the socio-economic environment is assumed to reduce the likelihood that single decision makers will consider all aspects of a decision making problem; consequently, numerous real world decision making processes occur in multi-person settings, so this investigation designs a multi-person decision making approach based on the constructed consistent complete fuzzy preference relations, which fuses individual preferences into collective ones and aggregates the overall information regarding each decision alternative to rank alternatives and select the optimal one(s). Finally, an illustrative example is presented to verify the developed approach.

2 Decision Making Using the Preference Relations

Most decision processes are known to be based on preference relation because it is a useful tool for modelling decision processes, particularly when aggregating the preferences of experts to produce group preferences[4,8,9]. Herrera et al. [11] proposed consistent fuzzy preference relations in accordance with two preference relations, namely multiplicative preference relation and fuzzy preference relations [12,13,14].

(1) Multiplicative preference relation.
The preferences of experts regarding a set of alternatives X can be denoted via a preference relation matrix $A \subset X \times X$, $A = (a_{ij})$, $a_{ij} \in [\frac{1}{9}, 9]$, where a_{ij} denotes the ratio of the preference degree

of alternative x_i over x_j. Since $a_{ij} = 1$ indicates indifference between x_i and x_j, $a_{ij} = 9$ indicates that x_i is strongly preferred to x_j. A is assumed to be a multiplicative reciprocal, namely

$$a_{ij} \cdot a_{ji} = 1 \tag{1}$$

Definition 1. A reciprocal multiplicative preference relation $A = (a_{ij})$ is consistent if

$$a_{ij} \cdot a_{jk} = a_{ik} \quad \forall i, j, k = 1,...,n. \tag{2}$$

(2) Fuzzy preference relation.
Expert preferences regarding a set of alternatives X are denoted via a positive preference relation matrix $P \subset X \times X$, with membership function: $\mu_p : X \times X \to [0,1]$, where $\mu_p(x_i, x_j) = p_{ij}$ indicates the ratio of the preference intensity of alternative x_i to that of x_j. If $p_{ij} = \frac{1}{2}$ implies indifference between x_i and x_j ($x_i \sim x_j$), $p_{ij} = 1$ indicates x_i is absolutely preferred to x_j, $p_{ij} = 0$ indicates x_j is absolutely preferred to x_i, and $p_{ij} > \frac{1}{2}$ indicates that x_i is preferred to x_j ($x_i > x_j$). P is assumed to be an additive reciprocal, given by

$$p_{ij} + p_{ji} = 1 \tag{3}$$

Proposition 1. Assume the existence of a set of alternatives, $X = \{x_1, x_2,.., x_n\}$, which is associated with a reciprocal multiplicative preference relation $A = (a_{ij})$, with $a_{ij} \in [\frac{1}{9}, 9]$. The corresponding reciprocal fuzzy preference relation, $P = (p_{ij})$ with $p_{ij} \in [0,1]$, associated with A is then given as follows:

$$p_{ij} = g(a_{ij}) = \tfrac{1}{2} \cdot (1 + \log_9 a_{ij}) \tag{4}$$

Proposition 2. Reciprocal additive fuzzy preference relations

$$p_{ij} + p_{jk} + p_{ki} = \tfrac{3}{2} \quad \forall i, j, k \tag{5}$$

$$p_{ij} + p_{jk} + p_{ki} = \tfrac{3}{2} \quad \forall i < j < k \tag{6}$$

$$p_{i(i+1)} + p_{(i+1)(i+2)} + ... + p_{(j-1)j} + p_{ji} = \tfrac{j-i+1}{2} \quad \forall i < j \tag{7}$$

Notably, according to Proposition 2, constructing consistent fuzzy preference relations only requires $n-1$ ($\{p_{12}, p_{23},..., p_{n-1n}\}$) judgments; the other incomplete elements can be done through additive transitivity. If the preference matrix contains values outside the interval $[0,1]$, namely within the interval $[-a, 1+a]$, a linear transform is required to preserve the reciprocity and additive transitivity, that is $f : [-a, 1+a] \to [0,1]$. The function is

$$f(x) = \frac{x+a}{1+2a} \tag{8}$$

For more details can refer to the study of Herrera et al. [11].

The method of Herrera et al. [11], can improve the performance of decision processes by reducing comparison times, but the construction of a consistent preference relation is restricted by the set of $n-1$ values $\{p_{12}, p_{23},..., p_{n-1n}\}$. Therefore, for more convenience and flexibility, the following develops a simple and practical method for constructing a consistent complete fuzzy preference relation, based on the incomplete fuzzy preference relation with the least judgments (i.e. $n-1$ judgments), as follows:

Step 1. For a multi-person decision-making problem, let $D = \{d_1, d_2,..., d_m\}$ denote the set of decision makers, while $X = \{x_1, x_2,.., x_n\}$ represents a discrete set of alternatives. The decision maker $d_k \in D$ compares each pair of alternatives using the discrete term set U_k, where $U_k = (u_{ij}^k)_{n \times n}$, $u_{ij}^k \in [\frac{1}{9}, 9]$. An incomplete preference relation $U_k = (u_{ij}^k)_{n \times n}$ is then constructed with only $n-1$ judgments, where experts can choice any row, column or diagonal to compare.

Step 2. Utilize the known elements in U_k and Eq.(2) to determine all the unknown elements in U_k and thus derive a consistent and complete preference relation $A_k = (a_{ij}^k)_{n \times n}$.

Step 3. Utilize Eqs.(4) and (8) to translate the complete preference relation $A_k = (a_{ij}^k)_{n\times n}$ into the complete fuzzy preference relation $F_k = (f_{ij}^k)_{n\times n}$.

Step 4. Utilize the averaging operator

$$f_{ij} = \tfrac{1}{m}\cdot(f_{ij}^1 \oplus f_{ij}^2 \oplus f_{ij}^3 \oplus ... \oplus f_{ij}^m) \text{ for all } i,j \quad (9)$$

to fuse all the consistent complete fuzzy preference relations $F_k = (f_{ij}^k)_{n\times n}$ $(k=1,2,...,m)$ into a collective complete fuzzy preference relation $F = (f_{ij})_{n\times n}$.

Step 5. Utilize the averaging operator to fuse all the fuzzy preference degrees f_{ij} $(j=1,2,...,n)$ in the ith line of the F, and obtain the average f_i of the ith alternative over all the other alternatives.

Step 6. Rank all the alternatives x_i $(i=1,2,...,n)$ and select the optimal one(s) according to the values of f_i $(i=1,2,...,n)$.
Step 7. End.

3 Numerical Example

This section presents a decision-making problem involving the evaluation of five candidates x_i $(i=1,2,...,5)$. The problem involves four decision makers d_k $(k=1,2,...,4)$ who compare these five alternatives using the discrete term set U_k, where $U_k = (u_{ij}^k)_{n\times n}$, $u_{ij}^k \in [\frac{1}{9}, 9]$, and provide the following judgments:

$$d_1: u_{12}^1 = 3,\ u_{13}^1 = \tfrac{1}{4},\ u_{14}^1 = 5,\ u_{15}^1 = \tfrac{1}{5}$$
$$d_2: u_{31}^2 = 7,\ u_{32}^2 = \tfrac{1}{3},\ u_{34}^2 = 5,\ u_{35}^2 = \tfrac{1}{4}$$
$$d_3: u_{21}^3 = 3,\ u_{31}^3 = \tfrac{1}{5},\ u_{41}^3 = 5,\ u_{51}^3 = \tfrac{1}{3}$$
$$d_4: u_{12}^4 = 3,\ u_{23}^4 = 5,\ u_{34}^4 = \tfrac{1}{3},\ u_{45}^4 = \tfrac{1}{5}$$

Obtaining the best alternative(s) involves the following steps:

Step 1. Use Eq. (1) and the above information provided by d_k $(k=1,2,3,4)$ to derive the incomplete preference relations $U_k = (u_{ij}^k)_{n\times n}$, respectively, where "-" represents the unknown variable.

$$U_1 = (u_{ij}^1)_{5\times 5} = \begin{bmatrix} 1 & 3 & \frac{1}{4} & 5 & \frac{1}{5} \\ \frac{1}{3} & 1 & - & - & - \\ 4 & - & 1 & - & - \\ \frac{1}{5} & - & - & 1 & - \\ 5 & - & - & - & 1 \end{bmatrix}$$

$$U_2 = (u_{ij}^2)_{5\times 5} = \begin{bmatrix} 1 & - & \frac{1}{7} & - & - \\ - & 1 & 3 & - & - \\ 7 & \frac{1}{3} & 1 & 5 & \frac{1}{4} \\ - & - & \frac{1}{5} & 1 & - \\ - & - & 4 & - & 1 \end{bmatrix}$$

$$U_3 = (u_{ij}^3)_{5\times 5} = \begin{bmatrix} 1 & \frac{1}{3} & 5 & \frac{1}{5} & 3 \\ 3 & 1 & - & - & - \\ \frac{1}{5} & - & 1 & - & - \\ 5 & - & - & 1 & - \\ \frac{1}{3} & - & - & - & 1 \end{bmatrix}$$

$$U_4 = (u_{ij}^4)_{5\times 5} = \begin{bmatrix} 1 & 3 & - & - & - \\ \frac{1}{3} & 1 & 5 & - & - \\ - & \frac{1}{5} & 1 & \frac{1}{3} & - \\ - & - & 3 & 1 & \frac{1}{5} \\ - & - & - & 5 & 1 \end{bmatrix}$$

Step 2. Utilize the known elements in U_k $(k=1,2,3,4)$ and Eq. (2) to determine all the unknown elements in U_k $(k=1,2,3,4)$:

$$A_1 = (a_{ij}^1)_{5\times 5} = \begin{bmatrix} 1 & 3 & \frac{1}{4} & 5 & \frac{1}{5} \\ \frac{1}{3} & 1 & \frac{1}{12} & \frac{5}{3} & \frac{1}{15} \\ 4 & 12 & 1 & 20 & \frac{4}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{20} & 1 & \frac{1}{25} \\ 5 & 15 & \frac{5}{4} & 25 & 1 \end{bmatrix}$$

$$A_2 = (a_{ij}^2)_{5\times 5} = \begin{bmatrix} 1 & \frac{1}{21} & \frac{1}{7} & \frac{5}{7} & \frac{1}{28} \\ 21 & 1 & 3 & 15 & \frac{3}{4} \\ 7 & \frac{1}{3} & 1 & 5 & \frac{1}{4} \\ \frac{7}{5} & \frac{1}{15} & \frac{1}{5} & 1 & \frac{1}{20} \\ 28 & \frac{4}{3} & 4 & 20 & 1 \end{bmatrix}$$

$$A_3 = (a_{ij}^3)_{5\times 5} = \begin{bmatrix} 1 & \frac{1}{3} & 5 & \frac{1}{5} & 3 \\ 3 & 1 & 15 & \frac{3}{5} & 9 \\ \frac{1}{5} & \frac{1}{15} & 1 & \frac{1}{25} & \frac{3}{5} \\ 5 & \frac{5}{3} & 25 & 1 & 15 \\ \frac{1}{3} & \frac{1}{9} & \frac{5}{3} & \frac{1}{15} & 1 \end{bmatrix}$$

$$A_4 = (a_{ij}^4)_{5\times 5} = \begin{bmatrix} 1 & 3 & 15 & 5 & 1 \\ \frac{1}{3} & 1 & 5 & \frac{5}{3} & \frac{1}{3} \\ \frac{1}{15} & \frac{1}{5} & 1 & \frac{1}{3} & \frac{1}{15} \\ \frac{1}{5} & \frac{3}{5} & 3 & 1 & \frac{1}{5} \\ 1 & 3 & 15 & 5 & 1 \end{bmatrix}$$

$$a_{23}^1 = a_{21}^1 \cdot a_{13}^1 = \frac{1}{a_{12}^1} \cdot a_{13}^1 = \frac{a_{13}^1}{a_{12}^1}$$

$$a_{24}^1 = a_{21}^1 \cdot a_{14}^1 = \frac{a_{14}^1}{a_{12}^1}$$

$$a_{25}^1 = a_{21}^1 \cdot a_{15}^1 = \frac{a_{15}^1}{a_{12}^1}$$

$$a_{42}^1 = \frac{1}{a_{24}^1} \quad a_{52}^1 = \frac{1}{a_{25}^1}$$

$$a_{32}^1 = a_{31}^1 \cdot a_{12}^1 = \frac{a_{12}^1}{a_{13}^1} = \frac{1}{a_{23}^1}$$

$$a_{34}^1 = a_{31}^1 \cdot a_{14}^1 = \frac{a_{14}^1}{a_{31}^1} \quad a_{43}^1 = \frac{1}{a_{34}^1}$$

$$a_{45}^1 = a_{41}^1 \cdot a_{15}^1 = \frac{a_{15}^1}{a_{14}^1} \quad a_{54}^1 = \frac{1}{a_{45}^1}$$

$$\vdots$$

Step 3.Use Eqs. (4) and (8) to obtain the corresponding consistent complete fuzzy preference relations:

$$F_1 = (f_{ij}^1)_{5\times 5} = \begin{bmatrix} 0.5 & 0.6707 & 0.2847 & 0.7500 & 0.2500 \\ 0.3293 & 0.5 & 0.1140 & 0.5793 & 0.0793 \\ 0.7153 & 0.8860 & 0.5 & 0.9653 & 0.4653 \\ 0.2500 & 0.4207 & 0.0347 & 0.5 & 0 \\ 0.7500 & 0.9207 & 0.5347 & 1 & 0.5 \end{bmatrix}$$

$$F_2 = (f_{ij}^2)_{5\times 5} = \begin{bmatrix} 0.5 & 0.0432 & 0.2080 & 0.4495 & 0 \\ 0.9568 & 0.5 & 0.6648 & 0.9063 & 0.4568 \\ 0.7920 & 0.3352 & 0.5 & 0.7415 & 0.2920 \\ 0.5505 & 0.0937 & 0.2585 & 0.5 & 0.0505 \\ 1 & 0.5432 & 0.7080 & 0.9495 & 0.5 \end{bmatrix}$$

$$F_3 = (f_{ij}^3)_{5\times 5} = \begin{bmatrix} 0.5 & 0.3293 & 0.7500 & 0.2500 & 0.6707 \\ 0.6707 & 0.5 & 0.9207 & 0.4207 & 0.8413 \\ 0.2500 & 0.0793 & 0.5 & 0 & 0.4207 \\ 0.7500 & 0.5793 & 1 & 0.5 & 0.9207 \\ 0.3293 & 0.1587 & 0.5793 & 0.0793 & 0.5 \end{bmatrix}$$

$$F_4 = (f_{ij}^4)_{5\times 5} = \begin{bmatrix} 0.5 & 0.7028 & 1 & 0.7972 & 0.5 \\ 0.2972 & 0.5 & 0.7972 & 0.5943 & 0.2972 \\ 0 & 0.2028 & 0.5 & 0.2972 & 0 \\ 0.2028 & 0.4057 & 0.7028 & 0.5 & 0.2028 \\ 0.5 & 0.7028 & 1 & 0.7972 & 0.5 \end{bmatrix}$$

Step 4. Utilize Eq. (9) to fuse all the consistent complete fuzzy preference relations $F_k = (f_{ij}^k)_{n\times n}$ $(k = 1,2,...,4)$ into a collective and complete fuzzy preference relation $F = (f_{ij})_{n\times n}$.

$$F = (f_{ij})_{5\times 5} = \begin{bmatrix} 0.5 & 0.4365 & 0.5607 & 0.5617 & 0.3552 \\ 0.5635 & 0.5 & 0.6242 & 0.6252 & 0.4187 \\ 0.4393 & 0.3758 & 0.5 & 0.5010 & 0.2945 \\ 0.4383 & 0.3748 & 0.4990 & 0.5 & 0.2935 \\ 0.6448 & 0.5813 & 0.7055 & 0.7065 & 0.5 \end{bmatrix}$$

Step 5. Utilize the averaging operator to fuse all the fuzzy preference degrees f_{ij} $(j = 1,2,...,5)$ in the ith line of F to obtain the average f_i of the ith alternative over all the other alternatives.

$$x_1 = 0.4828, x_2 = 0.5463, x_3 = 0.4221, x_4 = 0.4211, x_5 = 0.627$$

Step 6. Rank all the alternatives x_i $(i = 1,2,...,5)$ and identify the optimal one(s) according to the values of f_i $(i = 1,2,...,5)$.

$$x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$$

thus, the best alternative is x_5.

4 Conclusions

This investigation has developed a simple and practical method, which utilizes the incomplete fuzzy preference relation involving the least judgments (that is, $n-1$ judgments) to construct a consistent complete fuzzy preference relation. The most notable characteristic of the proposed method is that it requires the least judgments provided by the decision maker to create a consistent complete fuzzy preference relation, and thus it can not only reduce the time pressure faced by the decision maker but also avoid the need to check the consistency of fuzzy preference relations.

The approach adopted in this investigation may represent a new method of solving group decision making problems in complex environments. Future works can further study incomplete fuzzy preferences with multi-criteria.

References:

[1]. F. Chiclana, F.Herrera, E. Herrera-Viedma, Integrating three Representation Models in Fuzzy Multipurpose Decision making Based on Fuzzy Preference Relations, *Fuzzy Sets and Systems*, Vol.97, 1998, pp. 33-48.

[2]. T. Y. Choi, J. L. Hartley, An Exploration of Supplier Selection Practices across the Supply Chain, *Journal of Operations Management,* Vol.14, 1996, pp. 333-343.

[3]. Z. P. Fan, J. Ma, Q. Zhang, An Approach to Multiple Attribute Decision Making based on Fuzzy Preference Information on Alternatives, *Fuzzy Sets and Systems,* Vol.131, 2002, pp. 101-106.

[4]. J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer, Dordrecht, 1994.

[5]. T. Y. Hsieh, S. T. Lu, G. H. Tzeng, Fuzzy MCDM Approach for Planning and Design Tenders Selection in Public Office Buildings, *International Journal of Project Management* ,Vol.22, 2004, pp. 573-584.

[6]. H. Kerzner, A System Approach to Planning Scheduling and Controlling – Project Management, New York: Van Nostrand Reinhold, 1989, pp. 759-764.

[7]. F. H. Liu, H. L. Hai, The Voting Analytic Hierarchy Process Method for Selecting Supplier, *International Journal of Production Economics,* Vol.97, 2005, pp. 308-317.

[8]. T. L. Saaty, The Hierarchy Process, McGraw-Hill, New York, 1980.

[9]. T. Tanino, Fuzzy Preference Orderings in Goup Decision Making, *Fuzzy Sets and Systems*, Vol.12, 1984, pp. 117-131.

[10]. E. Triantaphyllou, Multi-criteria Decision Making Methods: a Comparative Study, Kluwer Academic Publishers, Dordrecht, 2000.

[11]. E. Herrera-Viedma, F. Herrera, F. Chiclana, M. Luque, Some Issues on Consistency of Fuzzy Preference Relations, *European Journal of Operational Research*, Vol.154, 2004, pp. 98-109.

[12]. T. C. Wang, Y. H. Chen, Applying Consistent Fuzzy Preference Relations to Partnership Selection, *Omega*, Vol.35, No.4, 2007, pp.384-388.

[13]. T. C. Wang, Y. H. Chen, Comparing Fuzzy Preference Relations and AHP as for the Decision Making, Proceedings of the 13th National Conference on Fuzzy Theory and Its Applications, Kaohsiung, Taiwan, 2005.

[14]. T. C. Wang, Y. H. Chen, Some Issues on Consistency of Fuzzy Analytic Hierarchy Process, Proceedings of 2006 International Conference on Machine Learning and Cybernetics, Dalian, China, 2006, pp. 1818-1822.

[15]. Y. M. Wang, C. Parkan, Multiple Attribute Decision Making Based on Fuzzy Preference Information on Alternatives: Ranking and Weighting, *Fuzzy Sets and Systems,* Vol.153, 2005, pp. 331-346.

[16]. C. C. Wei, C. F. Chien, M. J. Wang, An AHP-based Approach to ERP System Selection, *International Journal of Production Economics,* Vol. 96, 2005, pp. 47-62.